Pergamon

Int. J. Solids Structures Vol. 33, No. 20–22, pp. 3187–3214, 1996

0020–7683/96 $15.00 + .00

0020–7683(95)00247–2

ON FAILURE INDICATORS IN MULTI-DISSIPATIVE MATERIALS

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Abstract—Multi-dissipative constitutive descriptions of irreversible material degradation result in tangent operators that are made up of multiple rank-one updates of the elasticity tensors: multisurface elastoplasticity, plastic yielding combined with elastic degradation and multicrack models are representative examples. The spectral properties of these tangent operators determine failure conditions at the material level in terms of loss of uniqueness and discontinuous bifurcation of the incremental response. These failure properties are analysed herein by studying the eigensolution of the sum of n rank-one $(m \times m)$ matrices, and the eigensolution of n rank-one updates of the $(m \times m)$ identity matrix, whereby the tangent material tensors are written in matrix form. Analytical eigensolutions are presented and interpreted mechanically in terms of continuous and discontinuous failure indicators. In the light of these spectral properties, the failure indicators of single-dissipative materials are revisited and explicit results are presented for double-dissipative models in the form of plasticity combined with elastic-damage. In particular, it is shown that the activation and interaction of two dissipation processes may destabilize the tangent operators beyond the state resulting from a single active mechanism. Copyright © 1996 Elsevier Science Ltd.

1. INTRODUCTION

Irreversible non-linear material behavior is usually the consequence of multiple sources of energy dissipation which are activated during the loading process. In the mathematical framework of a constitutive model this behavior may be described by combining basic dissipation processes according to their degree of coupling which is required for realistic physical predictions, while satisfying the dissipation inequalities of the second law of thermodynamics (Maugin, 1992; Hansen and Schreyer, 1994).

In the rate format of multiple inelasticity each dissipation process is described by an evolution law which is activated by a yield (loading) function, by a dissipation potential, and by an inelastic multiplier (borrowing the terminology of plasticity, even though diverse dissipation mechanisms cover also degradation of elastic moduli). The classical multi-dissipative slip system occurs in crystal elastoplasticity, where the underlying deformation mechanisms arise from the relative slip along planes of different orientations. The slip theory of plasticity proposed by Batdorf and Budiansky (1949), was formalized by Koiter (1953) for independent plastic mechanisms, and was subsequently generalized for interacting planes by many contributors on the theoretical front (Mandel, 1965; Hill, 1966, 1967; Maier, 1969, 1970; Sewell, 1972, 1973, 1974; Christoffersen and Hutchinson, 1979), as well as on the computational side (Simo *et al.*, 1988; Pramono and William 1989; Feenstra, 1993). The present framework of multi-dissipative mechanics encompasses the important case of plasticity combined with elastic degradation (Bažant and Kim, 1979; Dragon and Mróz, 1979; Yazdani and Schreyer, 1990; Hansen and Schreyer, 1994) and multicrack models (Carol *et al.*, 1993).

A single dissipation mode is the normal format in traditional elastoplasticity, elastic degradation (Carol *et al.*, 1994), smeared cracking [fixed vs rotating] (Guzina *et al.*, 1995) and elastoplastic coupling (Maier and Hueckel, 1979; Han and Chen, 1986; Simo and Ju, 1987; Ju, 1989). In single-dissipation models, the spectral properties of the tangent operators have been extensively studied in order to determine failure conditions and failure modes at the material level in terms of continuous bifurcation (loss of uniqueness) and discontinuous bifurcation (localization) of the incremental response. Continuous failure is manifested by a singularity of the fourth order tangent stiffness tensor or its symmetric counterpart (Maier and Hueckel, 1979; Runesson and Mróz, 1989; Bigoni and Hueckel, 1991; Neilsen and Schreyer, 1993; Rizzi *et al.*, 1995). Discontinuous failure initiates when the second order localization tensor or its symmetric counterpart is singular (Rudnicki and Rice, 1975; Borré and Maier, 1989; Bigoni and Hueckel, 1991; Ottosen and Runesson, 1991a, 1991b; Neilsen and Schreyer, 1993; Rizzi *et al.*, 1995).

Originally, bifurcation studies of multisurface plasticity formulations were motivated by the overestimation of theoretical predictions based on smooth yield surface theories when compared to experimental observations in the area of plastic buckling (Sewell, 1972, 1973), localized necking and shear banding in metal sheets (Stören and Rice, 1975; Christoffersen and Hutchinson, 1979), and by the destabilizing aspects of non-associative flow in pressure-dependent materials (Rudnicki and Rice, 1975). The introduction of a vertex-type yield structure lowers the bifurcation predictions depending on the degree of coupling. This conclusion was already outlined by Mandel (1965) along the line of the propagation of acceleration waves in elastoplastic solids (Hill, 1962). Recent efforts on constitutive instabilities for multipotential theories confirmed this observation (Petryk, 1989, 1992; Ichicawa *et al.*, 1990; Aubry and Modaressi, 1992; Petryk and Thermann, 1992), even though a general framework for bifurcation analysis of tangent operators with arbitrary coupling, non-symmetric hardening and loss of normality is still lacking.

Material models with only one dissipation mechanism result in incremental relationships with tangent operators which have the form of a single rank-one modification (update) of the current elasticity tensor, while multi-dissipative models comprise multiple rank-one updates. In the present context of constitutive singularities, a rank-one update infers that a rank-one tensor, i.e. a tensor with only one non-zero eigenvalue, is added to or rather subtracted from a reference tensor. A multiple rank-one update means that n rank-one updates form the tensor modification, which may result in a rank-n modification, but generally it will differ from it. Although the problem of positive definiteness of symmetric rank-one and rank-two matrix updates (Brodlie *et al.*, 1973) has been extensively studied in the framework of secant methods for unconstrained minimization problems (Dennis and Schnabel, 1983), to the authors' knowledge there exist no analytic eigensolutions of multiple rank-one matrix updates. Moreover, while the symmetric rank-one eigenvalue problem is well understood, its non-symmetric counterpart is still the topic of on-going research (Jessup, 1993). Established results and new developments on the spectral properties of multiple rank-one updates relevant to this paper are presented in Appendix A.

In Sections 2 and 3, the paper develops a general setting for failure analysis of multi-dissipative materials at a given stage of the inelastic strain history. This stage exhibits n possible active dissipation modes (or mechanisms) within n_d mechanisms included in the material model. Both the tangent stiffness and the localization tensors are expressed by tensorial multiple (n) rank-one updates of the elasticity operators. These tangent operators are cast into matrix formats, whereby the fourth order tangent stiffness tensor is written as a (6×6) matrix and the second order localization tensor as a (3×3) matrix. In Appendix A, the paper examines the spectral properties of general $(m\times m)$ matrices obtained as the sum of n rank-one contributions, and the spectral properties of n rank-one modifications of the identity matrix of order m. The eigensolution leads to an associated eigenproblem which furnishes a consistent tool for numerical solutions and for analytical results when $n \leqslant 4$. The derivations are detailed in this paper for one and two rank-one updates ($n = 1, 2$), which include the formula for the perturbation of a determinant (Pearson, 1969).

These analytical developments provide a unified methodology for failure analysis of multi-dissipative constitutions in the above sense. The singularity conditions for single-dissipative materials are re-derived in Section 4 in the light of the new results, and are formalized in Section 5 with focus on double-dissipative materials combining plasticity with elastic degradation, allowing for non-associativity in stress and strain space. Thereby, the interaction of multi-dissipation processes destabilizes the tangent operators towards the singularity condition. Applications are presented in Section 6 for von Mises elastoplasticity coupled with scalar damage to illustrate the analytical results in uniaxial tension and in pure shear.

2. FLOW RULES FOR MULTI-DISSIPATIVE MATERIAL MODELS

The present study concerns the following constitutive relationship in terms of infinitesimal increments or rates, based on the additive strain rate decomposition into elastic and inelastic components:

$$\dot{\varepsilon} = \dot{\varepsilon}_E + \dot{\varepsilon}_I\,; \quad \dot{\varepsilon}_I = \sum_{\alpha=1}^{n} \dot{\varepsilon}_\alpha = \dot{\lambda}_\alpha \mathbf{m}_\alpha = \dot{\boldsymbol{\lambda}} \cdot \boldsymbol{m} \tag{1}$$

$$\dot{\mathbf{F}} = \frac{\partial \mathbf{F}}{\partial \boldsymbol{\sigma}} : \dot{\boldsymbol{\sigma}} + \frac{\partial \mathbf{F}}{\partial \mathbf{q}} \cdot \frac{\partial \mathbf{q}}{\partial \boldsymbol{\lambda}} \cdot \dot{\boldsymbol{\lambda}} = \boldsymbol{n} : \dot{\boldsymbol{\sigma}} - \mathbf{H} \cdot \dot{\boldsymbol{\lambda}} \leqslant \mathbf{0}\,; \quad \dot{\boldsymbol{\lambda}} \geqslant \mathbf{0}\,; \quad \dot{\mathbf{F}} \cdot \dot{\boldsymbol{\lambda}} = 0. \tag{2}$$

Reference is made here to strains and stresses in the small deformation regime; compact and index tensorial notations are adopted, and index summation convention is inferred. Dots over symbols mark time derivatives; single dots "·" between symbols denote inner products with single contraction, while colons ":" indicate inner products with double contraction; the operator "$\otimes$" designates the outer product.

Equation (1) expresses the additivity of elastic (recoverable) $\dot{\varepsilon}_E$ and inelastic (irreversible) $\dot{\varepsilon}_I$ strain rates, whereby the latter are the sum of n distinct dissipation mechanisms or yielding modes. The n possible active modes are a subset of all existing n_d mechanisms and may be activated in the initial state of stress $\boldsymbol{\sigma}$ and internal variables $\mathbf{q}$, such that $F_\alpha[\boldsymbol{\sigma}, \mathbf{q}] = 0$ ($\alpha = 1, 2, \ldots, n$) at the time t when the different dissipation processes do start. $\mathbf{F}$ is the vector of the loading functions which define the dissipation threshold, $F_\alpha = 0$, of the α^{th} dissipation mechanism with the elastic domain $F_\alpha \leqslant 0$. The dissipation factors $\boldsymbol{\lambda}$ measure the magnitudes of inelastic strain contributions in each dissipation mode λ_α; $\boldsymbol{m}$ defines the collection of flow directions $\mathbf{m}_\alpha$ (i.e. $m^\alpha_{kl} = (\mathbf{m}_\alpha)_{kl}$), and may be thought of as the stress gradient of a potential vector function $\mathbf{G}[\boldsymbol{\sigma}, \mathbf{q}]$. The functions in both $\mathbf{F}$ and $\mathbf{G}$ are assumed to be individually smooth in all situations considered.

In eqn (2) $\boldsymbol{n}$ collects the stress-gradients of the loading function $\mathbf{F}$, $\mathbf{H}$ is the matrix of hardening moduli $H_{\alpha\beta} = -\partial F_\alpha / \partial \lambda_\beta$, and $\dot{\mathbf{F}} \leqslant \mathbf{0}$ expresses the Prager's consistency condition for multiple dissipation processes. In general the gradients $\mathbf{n}_\alpha$ and $\mathbf{m}_\alpha$ are not linearly independent, but the assumption of linear independence (customary in multisurface plasticity), which implies full rank of $\boldsymbol{n}$ and $\boldsymbol{m}$, can lead to special properties which are of particular interest, as will be shown later. For a given $\dot{\boldsymbol{\sigma}}$, eqns (2) form a linear complementarity problem in the variables $\dot{\mathbf{F}}$, $\dot{\boldsymbol{\lambda}}$, associated to matrix $\mathbf{H}$.

Equations (1) and (2) describe the dissipation at a corner (i.e. in a singular, non-smooth point of the boundary of the current elastic region). They are linear in the sense that the flow directions $\mathbf{m}_\alpha$ do not depend on increments; they are non-associative when $\mathbf{m}_\alpha \neq \mathbf{n}_\alpha$; they are non-reciprocal if $H_{\alpha\beta} \neq H_{\beta\alpha}$ (reciprocal hardening makes $\mathbf{H}$ symmetric). The multiple dissipation setting may be that of classical plasticity theory (from the jargon of which some of the current terminology is borrowed), or a more general context which covers and possibly couples plasticity and elastic degradation.

Elastoplastic incremental laws in the form (1) and (2) were systematically studied primarily with regard to existence and uniqueness of the direct strain–stress rate $\dot{\varepsilon}[\dot{\boldsymbol{\sigma}}]$ and the inverse stress–strain rate $\dot{\boldsymbol{\sigma}}[\dot{\varepsilon}]$ relationships (Maier, 1969, 1970). For the present purpose, we summarize below some remarks and earlier results concerning eqns (1) and (2) (Mandel, 1965; Maier, 1969, 1970; Sewell, 1973).

2.1. *Remarks*

(a) If $\mathbf{E}$ denotes the current elastic stiffness tensor of the material with the usual symmetry and positive definite properties ($\mathbf{E} = \mathbf{E}_0$ in the absence of elastic moduli degradation), so that from eqn (1):

$$\dot{\boldsymbol{\sigma}} = \mathbf{E}:(\dot{\boldsymbol{\varepsilon}} - \dot{\boldsymbol{\lambda}} \cdot \boldsymbol{m}) \tag{3}$$

then the stress rate response $\dot{\boldsymbol{\sigma}}$ for prescribed $\dot{\boldsymbol{\varepsilon}}$ is governed by:

$$\dot{\mathbf{F}} = \boldsymbol{n}:\mathbf{E}:\dot{\boldsymbol{\varepsilon}} - \bar{\mathbf{H}} \cdot \dot{\boldsymbol{\lambda}} \leqslant \mathbf{0}; \quad \dot{\boldsymbol{\lambda}} \geqslant \mathbf{0}; \quad \dot{\mathbf{F}} \cdot \dot{\boldsymbol{\lambda}} = 0 \tag{4}$$

where

$$\bar{\mathbf{H}} \equiv \mathbf{H} - \mathbf{H}_c \quad \text{and} \quad \mathbf{H}_c \equiv -\boldsymbol{n}:\mathbf{E}:\boldsymbol{m}. \tag{5}$$

(b) There exists a unique $\dot{\boldsymbol{\varepsilon}}$ for any $\dot{\boldsymbol{\sigma}}$ if and only if $\mathbf{H}$ is a P-matrix (in the sense of Appendix B). When $\mathbf{H}$ is a P-matrix the flow rule of multi-dissipative materials is said to exhibit hardening; otherwise softening, except for $\mathbf{H} = 0$ when the behavior is perfectly plastic.

(c) If we assume linear independence for tensors $\mathbf{n}_\alpha$ and $\mathbf{m}_\alpha$, there exists a unique $\dot{\boldsymbol{\sigma}}$ for any $\dot{\boldsymbol{\varepsilon}}$ if and only if $\bar{\mathbf{H}}$ is a P-matrix. Here, matrix $\mathbf{H}_c$ in eqn (5) is termed the critical softening matrix because it corresponds to the critical softening parameter $H_c = -\mathbf{n}:\mathbf{E}:\mathbf{m}$ in the special case of single-dissipation (Maier and Hueckel, 1979). In fact, if there is no infinitesimal stress response for some kinematic perturbation $\delta\boldsymbol{\varepsilon}$, i.e. if $\bar{\mathbf{H}}$ is not a P-matrix, the material exhibits a snap-back or intrinsic instability (Maier, 1969). This occurrence is ruled out by assuming the P-behavior of $\bar{\mathbf{H}}$. This generalizes the well-known requirement that $\bar{H} > 0$ for single dissipation. Note that associativity implies that $\mathbf{H}_c$ is negative semidefinite, and negative definite if $\boldsymbol{m} = \boldsymbol{n}$ is a full-rank array, i.e. if $\mathbf{m}_\alpha = \mathbf{n}_\alpha$ are linearly independent.

(d) Lack of symmetry in $\mathbf{H}$ arises from non-reciprocal interaction between dissipation processes, in $\bar{\mathbf{H}}$ is either due to non-reciprocal interaction or to non-associativity. The assumption that $\bar{\mathbf{H}}$ is a P-matrix implies that the hardening matrix $\mathbf{H}$ has diagonal elements which satisfy the inequalities $H_{(\alpha)(\alpha)} > H^c_{(\alpha)(\alpha)}$ ($\alpha = 1, 2, \ldots, n$; ( ) indicates no sum on the repeated indices). This means that the direct hardening/softening coefficients must be algebraically larger than the critical values.

Other properties of the incremental responses in multi-dissipative materials were discussed in the context of plastic wave propagation (Mandel, 1965) and in the context of extremum properties (Maier, 1969). Within the present scope we are interested in finding an expression of the tangent operators of the material response, such that the pertinent failure indicators can be evaluated and analysed. To this end we consider:

(e) Tangent compliance.

Let $n' \leqslant n \leqslant n_d$ dissipation modes be active (in the sense that no unloading occurs) in a process mobilized by the given stress increment $\dot{\boldsymbol{\sigma}}$ where $\boldsymbol{n}'$, $\boldsymbol{m}'$, $\mathbf{H}'$ denote the arrays of the relevant gradients and the hardening submatrix, respectively. Then, provided that $\det[\mathbf{H}'] \neq 0$, the consistency condition of the active dissipation processes, $\dot{\mathbf{F}}' = 0$ in eqn (2), defines the active inelastic multipliers for stress control:

$$\dot{\boldsymbol{\lambda}}' = \mathbf{H}'^{-1} \cdot \boldsymbol{n}':\dot{\boldsymbol{\sigma}} \Leftrightarrow \dot{\lambda}'_\alpha = (\mathbf{H}'^{-1})_{\alpha\beta} (\mathbf{n}'_\beta)_{kl} \dot{\sigma}_{kl}. \tag{6}$$

The tangent compliance operator of the rate process, $\dot{\boldsymbol{\varepsilon}} = \mathbf{C}'_t:\dot{\boldsymbol{\sigma}}$, is then

$$\mathbf{C}'_t = \mathbf{C} + \mathbf{H}'^{-1}:(\boldsymbol{m}' \otimes \boldsymbol{n}') \Leftrightarrow C'^t_{ijkl} = C_{ijkl} + (\mathbf{H}'^{-1})_{\alpha\beta} m'^\alpha_{ij} n'^\beta_{kl}. \tag{7}$$

The tangent compliance results from $(n' \times n')$ rank-one updates of the current elastic compliance tensor $\mathbf{C} = \mathbf{E}^{-1}$, where only n' of them are independent contributions. In fact, defining the array $\check{\boldsymbol{n}}' = \mathbf{H}'^{-1} \cdot \boldsymbol{n}'$, the tangent compliance involves n' rank-one updates of the elastic compliance:

$$\mathbf{C}'_t = \mathbf{C} + \mathbf{m}'_\alpha \otimes \check{\mathbf{n}}'_\alpha. \tag{8}$$

(f) Tangent stiffness.
In full analogy to (e), the vector of inelastic multipliers for strain-control may be obtained from eqn (4), provided that $\det[\bar{\mathbf{H}}'] \neq 0$:

$$(\mathbf{H}' + \boldsymbol{n}' : \mathbf{E} : \boldsymbol{m}') \cdot \dot{\boldsymbol{\lambda}}' = \boldsymbol{n}' : \mathbf{E} : \dot{\boldsymbol{\varepsilon}} \;\Rightarrow\; \dot{\boldsymbol{\lambda}}' = \bar{\mathbf{H}}'^{-1} \cdot \boldsymbol{n}' : \mathbf{E} : \dot{\boldsymbol{\varepsilon}}. \tag{9}$$

The tangent stiffness operator of the rate process, $\dot{\boldsymbol{\sigma}} = \mathbf{E}'_t \cdot \dot{\boldsymbol{\varepsilon}}$, is then

$$\mathbf{E}'_t = \mathbf{E} - \bar{\mathbf{H}}'^{-1} : (\mathbf{E} : \boldsymbol{m}' \otimes \boldsymbol{n}' : \mathbf{E}) \;\Leftrightarrow\; E''_{ijkl} = E_{ijkl} - (\bar{\mathbf{H}}'^{-1})_{\alpha\beta} E_{ijpq} m'^{\alpha}_{pq} n'^{\beta}_{rs} E_{rskl} \tag{10}$$

or alternatively,

$$\mathbf{E}'_t = \mathbf{E} + \bar{\mathbf{H}}'^{-1} : (\bar{\boldsymbol{m}}' \otimes \bar{\boldsymbol{n}}') \quad \text{where} \quad \bar{\boldsymbol{n}}' \equiv \mathbf{E} : \boldsymbol{n}'; \quad \bar{\boldsymbol{m}}' \equiv -\mathbf{E} : \boldsymbol{m}'. \tag{11}$$

In compact form, using the notation $\hat{\boldsymbol{n}}' \equiv \bar{\mathbf{H}}'^{-1} \bar{\boldsymbol{n}}'$, i.e. $\hat{\mathbf{n}}'_\alpha \equiv \bar{H}'^{-1}_{\alpha\beta} \bar{\mathbf{n}}'_\beta$:

$$\mathbf{E}'_t = \mathbf{E} + \bar{\mathbf{m}}'_\alpha \otimes \hat{\mathbf{n}}'_\alpha. \tag{12}$$

Equation (12) shows that the resulting tangent stiffness involves n' rank-one updates of the elastic operator $\mathbf{E}$.

(g) Symmetry of $\mathbf{C}'_t$ and $\mathbf{E}'_t$.
In the general case of full coupling, whereby $\mathbf{H}'$ and $\bar{\mathbf{H}}'$ are non-diagonal and non-symmetric, the tangent operators are non-symmetric, even in the associative case ($\boldsymbol{m}' = \boldsymbol{n}', \bar{\boldsymbol{m}}' = -\bar{\boldsymbol{n}}'$). Symmetric operators are obtained only when all directions are parallel: in the compliance case when $\mathbf{m}'_\alpha = -\mathbf{n}'_\beta \,\forall \alpha, \beta$ such that $H'^{-1}_{\alpha\beta} \neq 0$, or $\mathbf{m}'_\alpha = \check{\mathbf{n}}'_\alpha \,\forall \alpha$; in the stiffness case when $\bar{\mathbf{m}}'_\alpha = -\bar{\mathbf{n}}'_\beta \,\forall \alpha, \beta$ such that $\bar{H}'^{-1}_{\alpha\beta} \neq 0$, or $\bar{\mathbf{m}}'_\alpha = -\hat{\mathbf{n}}'_\alpha \,\forall \alpha$. Associative flow rules result in symmetric tangent operators if $\mathbf{H}'$ and $\bar{\mathbf{H}}'$ are symmetric (Petryk, 1989), since $\bar{H}'^{-1}_{\alpha\beta} \bar{n}'^{\alpha}_{ij} \bar{n}'^{\beta}_{kl} = \bar{H}'^{-1}_{\alpha\beta} \bar{n}'^{\alpha}_{kl} \bar{n}'^{\beta}_{ij}$.

(h) Elastic isotropy.
If the elastic stiffness and compliance relations are isotropic, then the n' rank-one updates in eqns (8) and (12) turn out to be $(n'+1)$ rank-one updates of a tensor proportional to the fourth order symmetric (major and minor symmetries) identity tensor $\mathbf{i}^s_4$ (Rizzi, 1995), where $(\mathbf{i}^s_4)_{ijkl} = (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})/2$ ($\delta_{ij} = 1$ if $i = j$; $\delta_{ij} = 0$ if $i \neq j$). In fact, the isotropic elastic stiffness operator may be written $\mathbf{E} = 2\mu \mathbf{i}^s_4 + \lambda \mathbf{i}_2 \otimes \mathbf{i}_2$, where λ and μ denote the current values of Lame's moduli and $\mathbf{i}_2$ is the second order identity tensor, $(\mathbf{i}_2)_{ij} = \delta_{ij}$. Clearly the second term $\lambda \mathbf{i}_2 \otimes \mathbf{i}_2$ is a rank-one update of the first term $2\mu \mathbf{i}^s_4$.

(i) Strain-based formulation.
The same expression of the tangent operator (12) may be derived in a dual strain-based framework as an extension of the single-dissipative theory (see e.g. Carol *et al.*, 1994), where the loading functions are defined in the strain space as $\mathbf{F} = \mathbf{F}[\boldsymbol{\varepsilon}, \bar{\mathbf{q}}]$ (Petryk, 1989; Rizzi, 1995). By analogy with the strain decomposition (1) the inelastic stress increment is decomposed into:

$$\dot{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}}_E + \dot{\boldsymbol{\sigma}}_I; \quad \dot{\boldsymbol{\sigma}}_I = \sum_{\alpha=1}^{n} \dot{\boldsymbol{\sigma}}_\alpha = \dot{\lambda}_\alpha \bar{\mathbf{m}}_\alpha = \dot{\boldsymbol{\lambda}} \cdot \bar{\boldsymbol{m}} \tag{13}$$

where $\dot{\boldsymbol{\sigma}}_E = \mathbf{E} : \dot{\boldsymbol{\varepsilon}}$ and $\dot{\boldsymbol{\sigma}}_\alpha = -\mathbf{E} : \dot{\boldsymbol{\varepsilon}}_\alpha$ (which implies $\bar{\boldsymbol{m}} = -\mathbf{E} : \boldsymbol{m}$).

The consistency condition analogous to (2) involves $\dot{\mathbf{F}} = \bar{\boldsymbol{n}} : \dot{\boldsymbol{\varepsilon}} - \bar{\mathbf{H}} \cdot \dot{\boldsymbol{\lambda}} \leqslant \mathbf{0}$, whereby the array $\bar{\boldsymbol{n}}$ encloses the gradients of the loading surfaces in the strain space, and the matrix $\bar{\mathbf{H}}$

contains the hardening parameters $\bar{H}_{\alpha\beta} = -\partial F_\alpha / \partial \lambda_\beta$. It can be shown that $\bar{\boldsymbol{n}} = \boldsymbol{n} : \mathbf{E}$ and $\bar{\mathbf{H}} = \mathbf{H} + \boldsymbol{n} : \mathbf{E} : \boldsymbol{m}$ which gives a physical meaning to the barred quantities previously introduced.

(j) Coupling/decoupling.

The dependence of internal variables on the inelastic multipliers is in general fully coupled $\mathbf{q} = \mathbf{q}[\boldsymbol{\lambda}]$ (Mandel, 1965), or $\bar{\mathbf{q}}[\boldsymbol{\lambda}]$. Assuming uncoupling and one-to-one correspondence, then $q_\alpha = q_\alpha[\lambda_\alpha]$ (non-interacting modes; Koiter, 1953) or $\bar{q}_\alpha = \bar{q}_\alpha[\lambda_\alpha]$ (in the limit $q_\alpha = \bar{q}_\alpha = \lambda_\alpha$). Thus, decoupling of the dissipation mechanisms in stress or strain space results in diagonal hardening matrices $\mathbf{H}'$ or $\bar{\mathbf{H}}'$, respectively. However, full decoupling of the $\dot{\boldsymbol{\sigma}}[\dot{\boldsymbol{\varepsilon}}]$ response, when the system of the consistency conditions reduces to n' independent equations for the inelastic multipliers is achieved only when the effective hardening matrix $\bar{\mathbf{H}}'$ is diagonal, which requires that $H_{\alpha\beta} = H^c_{\alpha\beta}$ for $\alpha \neq \beta$.

3. FAILURE INDICATORS FOR MULTI-DISSIPATIVE MATERIALS

First, we define without specific references (which may be found e.g. in Bažant and Cedolin, 1991; Bigoni and Hueckel, 1991; Neilsen and Schreyer, 1993; Rizzi *et al.*, 1995; regarding the interrelationship between material stability, fracture theory and localization analysis) the nomenclature used below. Failure is here synonymous to bifurcation, i.e. the onset of alternative paths or material branching in the mechanical evolution of a homogeneous solid (or material specimen), which is originally in a state of uniform strain and stress state, and is conceivably unbounded in space (so that its boundary has a negligible role in its evolution). In this definition possible paths are itineraries, sequences of strain and stress states which fully comply with the adopted constitutive model, and with equilibrium (in the static and not in the thermodynamic sense), and geometric compatibility (internal, requiring displacement continuity).

Failure (or bifurcation) in the above sense is said to be continuous (or diffuse) when the homogeneity of strains and stresses is preserved in the solid specimen. Discontinuous failure (or localized failure) is understood as a bifurcation phenomenon which entails a spatial discontinuity in the strain rate field (strain localization), i.e. loss of strain and stress homogeneity, which usually occurs at the beginning of a fracture process. A failure indicator will be a scalar measure apt to quantify in some sense the distance between a given state and a failure event marked by a particular critical value (e.g. 0, while 1 is its value in the original, undamaged and unyielded state).

3.1. *Continuous failure (loss of stability) indicators*

Assume, along the evolution of the material specimen, the onset of a singular tangent stiffness $\mathbf{E}_t'$, eqn (12), for some strain rate $\dot{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{\varepsilon}}_{cf} \neq \mathbf{0}$:

$$\det[\mathbf{E}_t'] = 0, \quad \text{i.e.:} \quad \dot{\boldsymbol{\sigma}} = \mathbf{E}_t' : \dot{\boldsymbol{\varepsilon}}_{cf} = \mathbf{0}. \tag{14}$$

Such a stationary stress situation, of which an obvious illustrative example is the peak stress in a uniaxial tension test, can be thought of as bifurcation inasmuch as neighboring kinematic configurations are concerned which correspond to the same stress state. However, a concomitant meaningful mechanical characterization is the loss of stability according to the (strict) stability criterion of zero second-order work, i.e.:

$$d^2W = \dot{\boldsymbol{\sigma}}[\dot{\boldsymbol{\varepsilon}}] : \dot{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{\varepsilon}} : \mathbf{E}_t' : \dot{\boldsymbol{\varepsilon}} > 0 \quad \forall \dot{\boldsymbol{\varepsilon}} \neq \mathbf{0}. \tag{15}$$

Clearly, on the basis of this criterion, the onset of instability, i.e. $d^2W = 0$ for some $\dot{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{\varepsilon}}_{scf} \neq \mathbf{0}$ (while >0 for all other $\dot{\boldsymbol{\varepsilon}} \neq \mathbf{0}$), may be due to an occurrence alternative to eqn (14):

$$\det[\mathbf{E}_t'^s] = 0, \quad \text{where:} \quad \mathbf{E}_t'^s = \tfrac{1}{2}(\mathbf{E}_t' + \mathbf{E}_t'^T). \tag{16}$$

Following the terminology used in failure mechanics (see e.g. Bigoni and Hueckel, 1991; Neilsen and Schreyer, 1993; Rizzi *et al.*, 1995), we call eqn (14) a weak condition of continuous failure and eqn (16) a strong condition. Note that the strong condition is sufficient and necessary for $d^2W = 0$, since a quadratic form vanishes if and only if its (necessarily symmetric) matrix is singular. In terms of failure diagnostics the weak condition turns out to be less restrictive than the strong one.

It is worth stressing that the direction of the strain rate tensor $\dot{\varepsilon}$ defines, through the consistency condition (4) the n' active modes in the incremental process among the n possible active dissipation modes. This circumstance (reminded of by primes on the symbols of all affected quantities) implies the $\dot{\varepsilon}$-dependence of the tangent stiffness in eqn (12).

On the basis of the above preliminaries, it is natural to adopt for diffuse material failure the following definitions for weak and strong indicators, respectively:

$$e = \min_{\dot{\varepsilon}} \left\{ \frac{\det[\mathbf{E}_t'[\dot{\varepsilon}]]}{\det[\mathbf{E}]} \right\}; \quad e_s = \min_{\dot{\varepsilon}} \left\{ \frac{\det[\mathbf{E}_t'^s[\dot{\varepsilon}]]}{\det[\mathbf{E}]} \right\}. \tag{17}$$

3.2. *Discontinuous failure (localization) indicators*

The analysis of discontinuous failure originated from the arguments of propagation of acceleration waves in elastoplastic solids (Hill, 1962). Weak discontinuities of the second order in the continuum field are hypothesized across a discontinuity surface defined by the normal $\mathbf{N}$ in each point. Maxwell compatibility conditions require that the non-zero jump of the velocity gradient and of the strain rate across the surface must be of the form (see e.g. Ottosen and Runesson, 1991b)

$$[\![\nabla\dot{\mathbf{u}}]\!] = \dot{\gamma}\mathbf{M}\otimes\mathbf{N} \;\Rightarrow\; [\![\dot{\varepsilon}]\!] = \dot{\gamma}\tfrac{1}{2}(\mathbf{M}\otimes\mathbf{N}+\mathbf{N}\otimes\mathbf{M}) \tag{18}$$

where $\mathbf{M}$ is the unit vector which defines the direction of motion (polarization direction in wave propagation), and where $\dot{\gamma} \neq 0$ defines the unknown amplitude of the jump.

The condition of traction equilibrium across the surface, $[\![\dot{\mathbf{t}}]\!] = \mathbf{0}$, and the hypothesis of loading on both sides of the discontinuity lead to the definition of the localization tensor

$$\mathbf{Q}_t' = \mathbf{N}\cdot\mathbf{E}_t'\cdot\mathbf{N} \tag{19}$$

and to the weak localization condition

$$\mathbf{Q}_t'\cdot\mathbf{M}_{df} = \mathbf{0} \;\Rightarrow\; \det[\mathbf{Q}_t'] = 0. \tag{20}$$

On the other hand the condition for the propagation of the discontinuity leads to the definition of the so-called acoustic tensor $\mathbf{Q}_t'$ in $\mathbf{Q}_t'\cdot\mathbf{M} = (\rho c^2)\mathbf{M}$, where ρ is the mass density and c the wave speed. This means that the three wave speeds c are proportional to the eigenvalues of the acoustic tensor, while $\mathbf{M}$ are the associated eigenvectors. The condition for a stationary discontinuity is then obtained when the lowest eigenvalue ρc^2 vanishes.

A stronger condition for non-symmetric tangent operators may be expressed in terms of the localization energy, i.e. the quadratic form:

$$d^2W_{loc} = \dot{\gamma}^2\mathbf{M}_{sdf}\cdot\mathbf{Q}_t'\cdot\mathbf{M}_{sdf} = 0 \;\Rightarrow\; \det[\mathbf{Q}_t'^s] = 0. \tag{21}$$

This condition represents the second order work (15) when the localization kinematics $[\![\dot{\varepsilon}]\!]$ in the compatible form (18) are inserted.

The weak and strong indicators for localization may be defined, respectively, by expressions which are fully analogous to the ones for diffuse failure in eqn (17):

$$q = \min_{\dot{\varepsilon},\mathbf{N}} \left\{ \frac{\det[\mathbf{Q}'_t[\dot{\varepsilon},\mathbf{N}]]}{\det[\mathbf{Q}]} \right\}; \quad q_s = \min_{\dot{\varepsilon},\mathbf{N}} \left\{ \frac{\det[\mathbf{Q}'^{s}_t[\dot{\varepsilon},\mathbf{N}]]}{\det[\mathbf{Q}]} \right\}. \tag{22}$$

In our constitutive description the localization tensor derives from expression (12) of the tangent stiffness by contracting the orientation of the possible discontinuity surface for discontinuous bifurcation of $\dot{\varepsilon}$,

$$\mathbf{E}'_t = \mathbf{E} + \hat{\mathbf{m}}'_\alpha \otimes \hat{\mathbf{n}}'_\alpha \;\Rightarrow\; \mathbf{Q}'_t = \mathbf{N}\cdot\mathbf{E}'_t\cdot\mathbf{N} = \mathbf{Q} - \mathbf{a}'_\alpha \otimes \mathbf{d}'_\alpha \tag{23}$$

where $\mathbf{Q}$ is the elastic acoustic tensor and $\mathbf{a}'_\alpha = -\mathbf{N}\cdot\hat{\mathbf{m}}'_\alpha$, $\mathbf{d}'_\alpha = \hat{\mathbf{n}}'_\alpha\cdot\mathbf{N}$ are the traction vectors (Hill, 1962). Consequently, the localization tensor also entails n' rank-one updates of the elastic acoustic tensor.

3.3. *Remarks*

(a) In a uniaxial stress–strain diagram, zero compliance marks the onset of intrinsic or snap-back instability, i.e. loss of control on the specimen (and transition from a static to a dynamic phase) in a rigid displacement-imposing experimental rig. The counterpart of such an occurrence in the constitutive context of multi-dissipative material models formulated in Section 2, would be the singularity of at least one of the tangent compliances $\mathbf{C}'_t[\dot{\boldsymbol{\sigma}}]$ or of its symmetric part. Formally, considerations parallel to those expanded in Section 3.1 but concerning $\mathbf{C}'_t$, eqn (8), instead of $\mathbf{E}'_t$, eqn (12), would lead to indicators analogous to those defined by eqn (17). This topic is not considered here for brevity, also in view of its dubious mechanical meaning and its limited practical interest.

(b) The evaluation of the failure indicators is pursued here in closed form rather than numerically, in order to gain insight into the influence of various aspects. This is hardly possible for multi-dissipative materials ($n > 1$), as long as minimizations with respect to $\dot{\varepsilon}$ have to be carried out. Fortunately, it is reasonable to expect that the minimum is reached when all possible active dissipation modes are active (i.e. for $n' = n$). This conjecture is also weakly corroborated by localization analysis for single-dissipative materials (where Hill's linear comparison solid turns out to be a conservative assumption, cf. e.g. Borré and Maier, 1989; Ottosen and Runesson, 1991a). Although its validity will be investigated elsewhere, the assumption $n' = n$ will be assumed in the sequel, thus dropping all primes from the symbols. Thus, the evaluation of the indicators of diffuse failure reduces to mere assessment of determinants, according to eqn (17) without minimization; the diagnosis of localized failure, in eqn (22), reduces to a minimization with respect to the unit vector $\mathbf{N}$ normal to the discontinuity plane. Instead of analytically computing, which would be counterproductive for the present non-numerical purposes, spectral analysis will be employed in what follows in combination with known and new results on rank-one updates of matrices (these results are presented in Appendix A as for their mathematical aspects).

(c) In both modes of continuous and discontinuous failure, the indicators associated with the symmetric part of the operators give stronger conditions because of the Bromwich bounds on the eigenvalues of non-symmetric matrices (Mirsky, 1955). For the analysis of failure indicators it is necessary to investigate the spectral properties of $\mathbf{E}_t$ and $\mathbf{Q}_t$ by solving the eigenproblems $\mathbf{E}_t : \mathbf{x} = \Omega\mathbf{x}$ and $\mathbf{Q}_t\cdot\mathbf{M} = \omega\mathbf{M}$. The singularity conditions can be analysed considering instead the generalized eigenvalue problems $\mathbf{E}_t : \mathbf{x} = \Omega^*\mathbf{E} : \mathbf{x}$ and $\mathbf{Q}_t\cdot\mathbf{M} = \omega^*\mathbf{Q}\cdot\mathbf{M}$ which are indeed the eigenproblems for $\mathbf{E}^{-1} : \mathbf{E}_t$ and $\mathbf{Q}^{-1}\cdot\mathbf{Q}_t$ useful for establishing the condition of the vanishing determinant. The failure indicators for continuous and discontinuous failure labeled with e and q, simply mean

$$e = \frac{\det[\mathbf{E}_t]}{\det[\mathbf{E}]}, \quad e_s = \frac{\det[\mathbf{E}^s_t]}{\det[\mathbf{E}]}; \quad q = \frac{\det[\mathbf{Q}_t]}{\det[\mathbf{Q}]}, \quad q_s = \frac{\det[\mathbf{Q}^s_t]}{\det[\mathbf{Q}]} \tag{24}$$

where e and q indicate weak failure, whereas e_s and q_s measure strong failure conditions.

We finally conclude that we have to analyse the following problems:

(i) Determine eigenvalues and eigenvectors of $\mathbf{E}_t$ and $\mathbf{Q}_t$, whereby these tensors are obtained by n rank-one updates of the elasticity operators $\mathbf{E}$ and $\mathbf{Q}$. If the elasticity operators possess an isotropic format, the task is simplified since the elastic tensors result in a rank-one update of the identity tensor. Then the tangent operators are obtained by $(n+1)$ rank-one updates of the identity tensor. The eigenspectrum may be analysed for the pertinent matrix representation according to the derivations in Appendix A. The same considerations apply to the eigenspectrum of the symmetrized operators $\mathbf{E}_t^s$ and $\mathbf{Q}_t^s$ which are given by $2n$ updates of the elasticity tensors.

(ii) Determine singularities of the tangent operators which may be detected from the generalized eigenproblem of the operators $\mathbf{E}^{-1}:\mathbf{E}_t$ and $\mathbf{Q}^{-1}\cdot\mathbf{Q}_t$. This eigenvalue problem is easier to solve since these tensors are obtained directly by n rank-one updates of an identity tensor independently of the form of the elastic stiffness which may be non-isotropic. Analogously, for the symmetrized operators the number of updates is increased to $2n$.

(d) Note that the singularity conditions for diffuse failure result in implicit equations for the hardening parameters $e[\mathbf{H}] = 0$ which span hyper-surfaces in the $H_{\alpha\beta}$ space. For the localization condition these hyper-surfaces are marked with the label $\mathbf{N}$, which means that localization is activated in the direction $\mathbf{N}$ by the values of hardening parameters satisfying $q^{\mathrm{N}}[\mathbf{H}] = 0$. On the other hand, for obtaining the earliest onset of localization it is necessary to detect first the direction $\mathbf{N}_{\mathrm{df}}$ which renders $q^{\mathrm{N}}[\mathbf{H}]$ minimum at a given $\mathbf{H}$, and then look at the vanishing condition of the minimum value $\tilde{q}[\mathbf{H}]$:

$$\tilde{q}[\mathbf{H}] = \min_{\mathbf{N}} \{q^{\mathrm{N}}[\mathbf{H}] \mid \|\mathbf{N}\| = 1\} = 0. \tag{25}$$

This gives again an implicit equation for the hardening parameters. In a standard single-dissipative material this procedure renders the same result as the usual maximization of the hardening parameter $H_{\mathrm{df}}^{\mathrm{N}}$ necessary for discontinuous failure in the direction $\mathbf{N}$. However, it is difficult to define and apply this later method in the multi-dissipative context since one deals with the earliest point of localization along an arbitrary path in the hardening space. The first bifurcation is not achieved in general for the maximum value of all parameters $H_{\alpha\beta}$ along the path. In other words, the maximization problem is not easily defined for general hardening, when all $H_{\alpha\beta}$ are free parameters.

4. SPECTRAL ANALYSIS OF TANGENT OPERATORS FOR SINGLE-DISSIPATION MODELS

This section concerns materials with just one ($n = 1$) dissipation mechanism, which may be interpreted either as classical plasticity, or as elastic stiffness degradation, or as a combination of both (elastoplastic coupling) governed by a single yield function (and a single plastic multiplier). The purpose of this discussion is to illustrate and verify the present approach in a familiar context, and to revisit known results and expound some new remarks.

4.1. *Analysis of the tangent stiffness tensor*

The tangent stiffness operator of a single-dissipation mechanism is a special case of eqn (12):

$$\mathbf{E}_t = \mathbf{E} + \frac{\bar{\mathbf{m}} \otimes \bar{\mathbf{n}}}{\bar{H}} \tag{26}$$

where $\bar{\mathbf{m}}$ is the direction of the inelastic stress increment; $\bar{\mathbf{n}}$ is the gradient of the loading surface in the strain space; $\bar{H} > 0$ denotes the effective hardening parameter and $\mathbf{E}$ the current elastic stiffness ($\mathbf{E} = \mathbf{E}_0$ for elastoplasticity). The quantities with barred symbols are defined in strain space and are related to the dual quantities in stress space by means of

$\bar{\mathbf{n}} = \mathbf{E}:\mathbf{n}$; $\bar{\mathbf{m}} = -\mathbf{E}:\mathbf{m}$; $\bar{H} = H - H_c$; $H_c = -\mathbf{n}:\mathbf{E}:\mathbf{m}$ (Carol *et al.*, 1994). In the associative case ($\mathbf{m} = \mathbf{n}$ or $\bar{\mathbf{m}} = -\bar{\mathbf{n}}$) the tangent operator becomes symmetric.

Let us consider first isotropic elastic behavior, for which the tangent stiffness operator, according to eqn (26), has the form

$$\mathbf{E}_t = 2\mu\mathbf{i}_4^s + \lambda\mathbf{i}_2 \otimes \mathbf{i}_2 + \frac{\bar{\mathbf{m}} \otimes \bar{\mathbf{n}}}{\bar{H}} = 2\mu\mathbf{i}_4^s + \mathbf{U}_2 \tag{27}$$

where λ and μ are the Lamé constants and $\mathbf{i}_4^s$ denotes the fourth order symmetric identity tensor. Clearly, $\mathbf{E}_t$ may be conceived as the modification of the reference shear stiffness $2\mu\mathbf{i}_4^s$ through two rank-one updates in the sense of Appendix A, represented by $\mathbf{U}_2$. According to the remarks in Appendix A, the eigensolution $\mathbf{U}_2:\mathbf{x} = u\mathbf{x}$ is tackled by considering eigentensors in the form $\mathbf{x} = \alpha_1\mathbf{i}_2 + \alpha_2\bar{\mathbf{m}}$ and, thus reduces to:

$$\begin{cases} (3\lambda - u) \quad \alpha_1 + \quad \lambda\,\mathrm{tr}\,[\bar{\mathbf{m}}] \quad \alpha_2 = 0 \\ \dfrac{\mathrm{tr}\,[\bar{\mathbf{n}}]}{\bar{H}} \quad \alpha_1 + \left(\dfrac{\bar{\mathbf{n}}:\bar{\mathbf{m}}}{\bar{H}} - u\right) \quad \alpha_2 = 0 \end{cases} \tag{28}$$

where $\mathrm{tr}[\mathbf{o}]$ is the trace of $\mathbf{o}$ ($\mathrm{tr}[\mathbf{o}] = \mathbf{i}_2:\mathbf{o}$). According to eqns (A.6) and (A.10) the eigenvalues $u_{1,2}$ of $\mathbf{U}_2$ in eqn (28) and the corresponding values $\Omega_{1,2}$ of $\mathbf{E}_t$ (interpreted as those of $2\mu\mathbf{i}_4^s$ modified by the updates $\mathbf{U}_2$) read, respectively:

$$u_{1,2} = \frac{1}{2}\left(\left(3\lambda + \frac{\bar{\mathbf{n}}:\bar{\mathbf{m}}}{\bar{H}}\right) \pm \sqrt{\left(3\lambda + \frac{\bar{\mathbf{n}}:\bar{\mathbf{m}}}{\bar{H}}\right)^2 - 4\left(3\lambda\frac{\bar{\mathbf{n}}:\bar{\mathbf{m}}}{\bar{H}} - \frac{\lambda\mathrm{tr}[\bar{\mathbf{n}}]\mathrm{tr}[\bar{\mathbf{m}}]}{\bar{H}}\right)}\right) \tag{29}$$

$$\Omega_{1,2} = 2\mu \pm u_{1,2}. \tag{30}$$

The limit-point condition $\det[\mathbf{E}_t] = 0$ reduces to the vanishing condition for the minimum eigenvalue $\Omega_2 = 2\mu + u_2 = 0$. Solving this equation with respect of $\bar{H}$ we obtain the hardening parameter $\bar{H}_{cf}$ which indicates continuous failure:

$$\bar{H}_{cf} = \frac{-(3\lambda + 2\mu)\bar{\mathbf{n}}:\bar{\mathbf{m}} + \lambda\,\mathrm{tr}\,[\bar{\mathbf{n}}]\,\mathrm{tr}\,[\bar{\mathbf{m}}]}{2\mu(3\lambda + 2\mu)} = -\bar{\mathbf{n}}:\mathbf{C}:\bar{\mathbf{m}}. \tag{31}$$

Here $\mathbf{C} = \mathbf{E}^{-1}$ denotes the isotropic elastic compliance:

$$\mathbf{C} = \frac{-\lambda}{2\mu(3\lambda + 2\mu)}\mathbf{i}_2 \otimes \mathbf{i}_2 + \frac{1}{2\mu}\mathbf{i}_4^s. \tag{32}$$

This result coincides with the one established by Rizzi *et al.* (1995) by assuming a critical strain rate $\dot{\boldsymbol{\varepsilon}}_{cf} = \mathbf{C}:\bar{\mathbf{m}}$ for deriving the singularity condition. This critical strain rate tensor is the eigentensor $\mathbf{x}$ associated to the zero eigenvalue of $\mathbf{E}_t$. In fact, from the solution of the system of two equations (28) for $u = -2\mu$, we obtain

$$\frac{\alpha_1}{\alpha_2} = \frac{-\lambda\,\mathrm{tr}\,[\bar{\mathbf{m}}]}{3\lambda + 2\mu} \tag{33}$$

which leads to an eigenvector $\mathbf{x}$ proportional to $\mathbf{C}:\bar{\mathbf{m}}$.

It is worth noting that the limit-point condition in terms of a stress-based formulation yields $H_{cf} = 0$ with $\mathbf{x} = \mathbf{m}$. This means that the strain rate tensor $\dot{\boldsymbol{\varepsilon}}_{cf}$ for vanishing stress rates has the direction of the inelastic strain rate tensor.

As expected from the results of two rank-one modifications of the identity tensor (Appendix A), in general two eigenvalues of $\mathbf{E}_t$ are modified with respect to the eigenvalues of $\mathbf{E}$ which may be complex in the case of non-associative flow rules resulting in non-symmetric updates. However, depending on the choice of the tensors $\bar{\mathbf{n}}$ and $\bar{\mathbf{m}}$, the two updates of $\mathbf{E}$ may modify only one of its eigenvalues or may cause no change of its eigenspectrum. This circumstance is closely related to the real-valued eigenvalues of $\mathbf{E}_t$, even for non-associative flow. Decomposing $\bar{\mathbf{n}}$ and $\bar{\mathbf{m}}$ into their deviatoric and volumetric parts ($\bar{\mathbf{n}}_d$, $\bar{\mathbf{m}}_d$ and $\bar{\mathbf{n}}_v$, $\bar{\mathbf{m}}_v$, respectively), so that $\bar{\mathbf{n}}:\bar{\mathbf{m}} = \bar{\mathbf{n}}_d:\bar{\mathbf{m}}_d+\bar{\mathbf{n}}_v:\bar{\mathbf{m}}_v$, it is worth noting that:

(a) If $\bar{\mathbf{n}}_v = \mathbf{0}$, $\bar{\mathbf{m}}_d = \mathbf{0}$ or $\bar{\mathbf{n}}_d = \mathbf{0}$, $\bar{\mathbf{m}}_v = \mathbf{0}$, i.e. one of the two tensors $\bar{\mathbf{n}}$, $\bar{\mathbf{m}}$ is purely deviatoric while the other one is purely volumetric, $\bar{\mathbf{n}}:\bar{\mathbf{m}} = 0$. Therefore, the eigenvalues of $\mathbf{U}_2$ in eqn (29) are $u_1 = 3\lambda$ and $u_2 = 0$. The eigenvalues of $\mathbf{E}_t$ coincide with those of $\mathbf{E}$, which are the five 2μ shear eigenvalues and the bulk eigenvalue $3K = 3\lambda+2\mu$, where K denotes the bulk modulus (Neilsen and Schreyer, 1993).

(b) If one of the two tensors is purely deviatoric, i.e. $\bar{\mathbf{n}}_v = \mathbf{0}$ or $\bar{\mathbf{m}}_v = \mathbf{0}$, from eqn (29) we have $u_1 = 3\lambda$ and $u_2 = \bar{\mathbf{n}}:\bar{\mathbf{m}}/\bar{H}$, if $3\lambda-\bar{\mathbf{n}}:\bar{\mathbf{m}}/\bar{H} \geqslant 0$, or, conversely, $u_1 = \bar{\mathbf{n}}:\bar{\mathbf{m}}/\bar{H}$ and $u_2 = 3\lambda$, if $3\lambda-\bar{\mathbf{n}}:\bar{\mathbf{m}}/\bar{H} \leqslant 0$. This means that the bulk eigenvalue $3K = 3\lambda+2\mu$ is not modified, while one shear eigenvalue is modified to $2\mu+\bar{\mathbf{n}}:\bar{\mathbf{m}}/\bar{H}$, which remains real-valued and which vanishes for $\bar{H} = -\bar{\mathbf{n}}:\bar{\mathbf{m}}/2\mu$.

(c) If one of the two tensors is purely volumetric, i.e. $\bar{\mathbf{n}}_d = \mathbf{0}$ or $\bar{\mathbf{m}}_d = \mathbf{0}$, since we have $\bar{\mathbf{n}}:\bar{\mathbf{m}} = \mathrm{tr}[\bar{\mathbf{n}}]\mathrm{tr}[\bar{\mathbf{m}}]/3$, eqn (29) yields: $u_1 = 3\lambda+\bar{\mathbf{n}}:\bar{\mathbf{m}}/\bar{H}$ and $u_2 = 0$, if $3\lambda+\bar{\mathbf{n}}:\bar{\mathbf{m}}/\bar{H} \geqslant 0$; or conversely $u_1 = 0$ and $u_2 = 3\lambda+\bar{\mathbf{n}}:\bar{\mathbf{m}}/\bar{H}$, if $3\lambda+\bar{\mathbf{n}}:\bar{\mathbf{m}}/\bar{H} \leqslant 0$. It turns out that only the bulk eigenvalue is modified to $3K+\bar{\mathbf{n}}:\bar{\mathbf{m}}/\bar{H}$, which remains real-valued and which vanishes for $\bar{H} = -\bar{\mathbf{n}}:\bar{\mathbf{m}}/3K$.

(d) If both $\bar{\mathbf{n}}$ and $\bar{\mathbf{m}}$ contain a volumetric, as well as a deviatoric part, two eigenvalues are modified which may be complex for non-associative flow rules depending on the sign of the discriminant under the square root in eqn (29).

The limit-point condition can be directly derived by using eqn (A.9) for the determinant of a rank-one update of the identity matrix. In fact, by considering the generalized eigenvalue problem $\mathbf{E}_t:\mathbf{x} = \Omega^*\mathbf{E}:\mathbf{x}$, we can write the continuous failure indicator e as

$$e = \frac{\det[\mathbf{E}_t]}{\det[\mathbf{E}]} = \det\left[\mathbf{i}_4^s - \frac{\mathbf{m}\otimes\bar{\mathbf{n}}}{\bar{H}}\right] = 1-\left(\frac{\mathbf{n}:\mathbf{E}:\mathbf{m}}{\bar{H}}\right) = \frac{H}{\bar{H}} = 0 \;\Rightarrow\; H_{cf} = 0. \tag{34}$$

This result holds for general non-isotropic elastic stiffness or compliance tensors, and it confirms the previous results in eqn (31) [$\bar{H}_{cf} = \mathbf{n}:\mathbf{E}:\mathbf{m} > 0$]. Since $\mathbf{n}:\mathbf{E}:\mathbf{m}$ is assumed to be positive, eqn (34) shows that $\bar{H} > H$ and $\det[\mathbf{E}_t] < \det[\mathbf{E}]$. Thus, locking is ruled out in the tangent response (Fig. 1).

The loss of stability condition, $d^2W = 0$, i.e. the singularity condition for the symmetric tangent operator $\mathbf{E}_t^s$ can be easily derived by similar reasoning. In fact, we have:

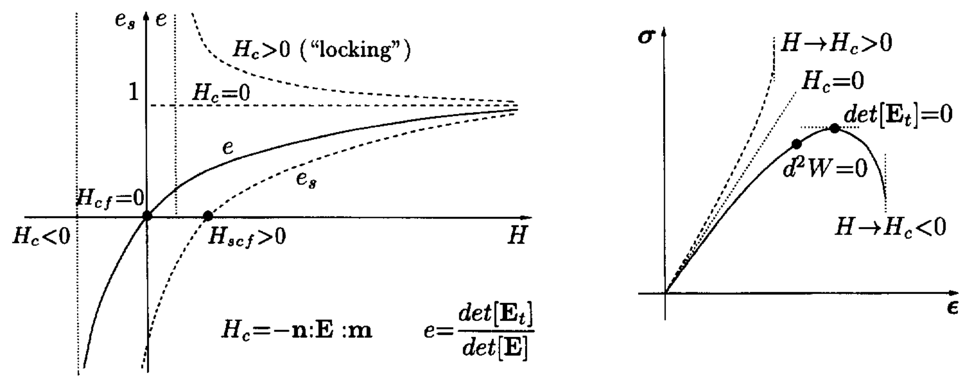

Fig. 1. Hierarchy of continuous failure indicators in single-dissipative materials.

$$\mathbf{E}_t^s = \mathbf{E} + \frac{1}{2}\left(\frac{\bar{\mathbf{m}} \otimes \bar{\mathbf{n}}}{\bar{H}} + \frac{\bar{\mathbf{n}} \otimes \bar{\mathbf{m}}}{\bar{H}}\right) \Rightarrow \mathbf{E}^{-1} : \mathbf{E}_t^s = \mathbf{i}_4^s - \frac{1}{2}\frac{\mathbf{m} \otimes \bar{\mathbf{n}}}{\bar{H}} + \frac{1}{2}\frac{\mathbf{n} \otimes \bar{\mathbf{m}}}{\bar{H}}. \quad (35)$$

Then, by using the perturbation formula (A.12), the normalized determinant of the symmetric tangent operator which results from two rank-one updates of the identity matrix, reads

$$e_s = \frac{\det[\mathbf{E}_t^s]}{\det[\mathbf{E}]} = \left(1 - \frac{1}{2}\frac{\mathbf{m} : \bar{\mathbf{n}}}{\bar{H}}\right)\left(1 + \frac{1}{2}\frac{\mathbf{n} : \bar{\mathbf{m}}}{\bar{H}}\right) + \frac{1}{2}\frac{\mathbf{m} : \bar{\mathbf{m}}}{\bar{H}}\frac{1}{2}\frac{\bar{\mathbf{n}} : \mathbf{n}}{\bar{H}} \quad (36)$$

$$e_s = 1 - \left(\frac{\mathbf{n} : \mathbf{E} : \mathbf{m}}{\bar{H}}\right) - \left(\frac{(\mathbf{n} : \mathbf{E} : \mathbf{n})(\mathbf{m} : \mathbf{E} : \mathbf{m}) - (\mathbf{n} : \mathbf{E} : \mathbf{m})^2}{4\bar{H}^2}\right). \quad (37)$$

The condition for a vanishing determinant thus becomes a second order equation for the hardening parameter $H = \bar{H} - \mathbf{n} : \mathbf{E} : \mathbf{m}$, which gives rise to two solutions:

$$H_{1,2} = \tfrac{1}{2}(-\mathbf{n} : \mathbf{E} : \mathbf{m} \pm \sqrt{(\mathbf{n} : \mathbf{E} : \mathbf{n})(\mathbf{m} : \mathbf{E} : \mathbf{m})}). \quad (38)$$

The positiveness of $\bar{H}$ entails softening parameters larger than the critical negative value: $H > H_c = -\mathbf{n} : \mathbf{E} : \mathbf{m}$. Clearly, the E-orthogonality condition $\mathbf{n} : \mathbf{E} : \mathbf{m} = 0$, which defines the boundary between positive and negative values of H_c (and which thus restricts the permissible degree of non-normality), does not coincide in general with the orthogonality condition $\mathbf{n} : \mathbf{m} = 0$ (Fig. 2).

By rescaling $\mathbf{m}$ and $\mathbf{n}$ it is always possible to set $\mathbf{n} : \mathbf{E} : \mathbf{n} = \mathbf{m} : \mathbf{E} : \mathbf{m} = r$ where r is a positive constant. Since $\mathbf{y} : \mathbf{E} : \mathbf{y} = r$ represents an ellipsoid in the space of the symmetric second order tensor $\mathbf{y}$ (Fig. 2), in view of the strict convexity of this ellipsoid, one can write $\mathbf{n} : \mathbf{E} : (\mathbf{n} - \mathbf{m}) > 0$ for any $\mathbf{n}$, $\mathbf{m}$ not proportional to each other (Maier and Hueckel, 1979).

Of the two roots in eqn (38), the lesser one, H_2, is a hardening modulus in the snap-back (or subcritical) range (since $H_2 \leqslant H_c$); the larger one, H_1, in the yielding process with monotonically decreasing hardening represents the value H_{scf} at which stability is lost first (in the sense of $d^2W = 0$):

$$H_{scf} = \tfrac{1}{2}(\sqrt{(\mathbf{n} : \mathbf{E} : \mathbf{n})(\mathbf{m} : \mathbf{E} : \mathbf{m})} - \mathbf{n} : \mathbf{E} : \mathbf{m}) \geqslant H_{cf} = 0. \quad (39)$$

This critical hardening modulus H_{scf} is found to be positive and zero only in the associative case. The loss of stability condition $d^2W = 0$ turns out to represent a more restrictive condition for continuous failure (it occurs earlier) than the loss of uniqueness of the incremental response, $\det[\mathbf{E}_t] = 0$, as indicated in the left sketch of Fig. 1, where e and e_s are plotted. This fact is due to the positive term which shows up in the second parenthesis in eqn (37), but which is missing in eqn (34). This destabilization of the determinant value should be expected in view of the Bromwich bounds on the eigenvalues of a non-symmetric

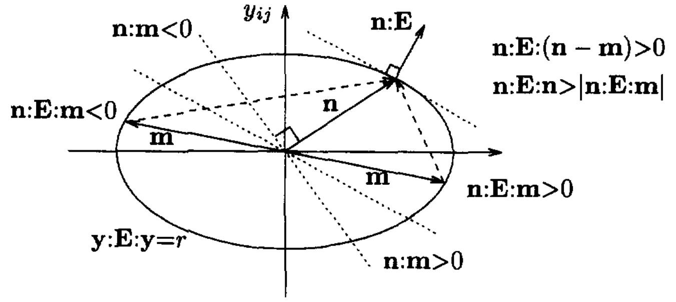

Fig. 2. Representation of quadratic and bilinear forms associated to E.

matrix. The above results were first established through different considerations by Maier and Hueckel (1979), and later by Runesson and Mróz (1989), and Neilsen and Schreyer (1993).

4.2. *Analysis of the tangent localization tensor*

The localization tensor may be developed either from the general expression by considering only one contribution in eqn (23), or directly by contracting the tangent stiffness (26) with the unit vector $\mathbf{N}$ normal to the possible discontinuity surface:

$$\mathbf{Q}_t = \mathbf{Q} - \frac{\mathbf{a} \otimes \mathbf{b}}{\bar{H}} \tag{40}$$

where $\mathbf{Q} = \mathbf{N} \cdot \mathbf{E} \cdot \mathbf{N}$ is the elastic acoustic tensor and $\mathbf{a} = -\mathbf{N} \cdot \bar{\mathbf{m}}$, $\mathbf{b} = \bar{\mathbf{n}} \cdot \mathbf{N}$ are the traction vectors. For associative material models $\bar{\mathbf{m}} = -\bar{\mathbf{n}}$ and, hence, $\mathbf{b} = \mathbf{a}$.

As for the tangent stiffness, the eigenspectrum of $\mathbf{Q}_t$ for isotropic elasticity, and the weak and strong localization conditions are derived using the results for rank-one updates of the identity matrix.

The generalized eigenvalue problem $\mathbf{Q}_t \cdot \mathbf{x} = \omega^* \mathbf{Q} \cdot \mathbf{x}$ is the eigensolution of a rank-one update of the identity matrix, since $\mathbf{Q}^{-1} \cdot \mathbf{Q}_t = \mathbf{i}_2 - \mathbf{Q}^{-1} \cdot \mathbf{a} \otimes \mathbf{b}/\bar{H}$. Thus, according to the results for rank-one updates in Appendix A, we have:

$$\omega_1^* = \omega_2^* = 1 \quad \omega_3^* = 1 - \frac{\mathbf{b} \cdot \mathbf{Q}^{-1} \cdot \mathbf{a}}{\bar{H}}. \tag{41}$$

The localization condition is reached when the discontinuous failure indicator vanishes ($q = 0$). This yields the critical hardening parameter for discontinuous failure for a given direction $\mathbf{N}$:

$$q = \frac{\det[\mathbf{Q}_t]}{\det[\mathbf{Q}]} = 1 - \left(\frac{\mathbf{b} \cdot \mathbf{Q}^{-1} \cdot \mathbf{a}}{\bar{H}}\right) = 0 \;\Rightarrow\; \bar{H}_{\mathrm{df}}^{\mathrm{N}} = \mathbf{b} \cdot \mathbf{Q}^{-1} \cdot \mathbf{a}. \tag{42}$$

This fully coincides with the established results by Ottosen and Runesson (1991a) for elastoplastic materials, and with those by Rizzi *et al.* (1995) for materials with elastic degradation.

The strong localization condition for vanishing second order work $d^2 W_{\mathrm{loc}} = 0$ involves the singularity of the symmetric tangent localization tensor $\mathbf{Q}_t^s$, and hence the determinant of the modification of the identity matrix by two rank-one updates. In fact

$$\mathbf{Q}_t^s = \mathbf{Q} - \frac{1}{2}\left(\frac{\mathbf{a} \otimes \mathbf{b}}{\bar{H}} + \frac{\mathbf{b} \otimes \mathbf{a}}{\bar{H}}\right) \;\Rightarrow\; \mathbf{Q}^{-1} \cdot \mathbf{Q}_t = \mathbf{i}_2 - \frac{1}{2}\frac{\tilde{\mathbf{a}} \otimes \mathbf{b}}{\bar{H}} - \frac{1}{2}\frac{\tilde{\mathbf{b}} \otimes \mathbf{a}}{\bar{H}} \tag{43}$$

where $\tilde{\mathbf{a}} = \mathbf{Q}^{-1} \cdot \mathbf{a}$ and $\tilde{\mathbf{b}} = \mathbf{Q}^{-1} \cdot \mathbf{b}$. From the perturbation formula eqn (A.12), and following the derivation of eqn (37) leads to:

$$q_s = \frac{\det[\mathbf{Q}_t^s]}{\det[\mathbf{Q}]} = 1 - \left(\frac{\mathbf{b} \cdot \mathbf{Q}^{-1} \cdot \mathbf{a}}{\bar{H}}\right) - \left(\frac{(\mathbf{a} \cdot \mathbf{Q}^{-1} \cdot \mathbf{a})(\mathbf{b} \cdot \mathbf{Q}^{-1} \cdot \mathbf{b}) - (\mathbf{b} \cdot \mathbf{Q}^{-1} \cdot \mathbf{a})^2}{4\bar{H}^2}\right). \tag{44}$$

The second parenthesis in eqn (44) is non-negative and gives an additional negative contribution which destabilizes the determinant when compared to the expression (42). As a result, $d^2 W_{\mathrm{loc}} = 0$ is a more restrictive condition for localization than $\det[\mathbf{Q}_t] = 0$ (which is satisfied earlier when H decreases monotonically), in agreement with the Bromwich bounds on the eigenvalues. In associative models, i.e. with $\mathbf{b} = \mathbf{a}$, the above term in parentheses vanishes and the two conditions coincide as expected.

Condition (44) for vanishing values of $\det[\mathbf{Q}_t^s]$ and indicator q_s results in a second order equation for $\bar{H}$ with the roots $\bar{H}_{1,2}$. The second root $\bar{H}_2$ is in the subcritical softening range, while the first one is positive and marks the strong critical threshold for discontinuous failure:

$$\bar{H}_{sdf}^{N} = \bar{H}_1 = \tfrac{1}{2}((\mathbf{b}\cdot\mathbf{Q}^{-1}\cdot\mathbf{a})+\sqrt{(\mathbf{a}\cdot\mathbf{Q}^{-1}\cdot\mathbf{a})(\mathbf{b}\cdot\mathbf{Q}^{-1}\cdot\mathbf{b})}) \geqslant \bar{H}_{df}^{N}. \tag{45}$$

Note that both the strong and weak localization conditions may be satisfied in the hardening range, since $H_{df}^{N} = \bar{H}_{df}^{N} - \mathbf{n}:\mathbf{E}:\mathbf{m}$ can be positive, negative or zero, even for positive values of $\bar{H}_{df}^{N}$.

The localization conditions (42) and (44) may be compared directly with the dual continuous failure conditions if we define the fourth tensor $\mathbf{E}^N$ which represents a projection of the elastic stiffness on to the direction $\mathbf{N}$ as follows:

$$\mathbf{E}^N = (\mathbf{E}\cdot\mathbf{N})\cdot\mathbf{Q}^{-1}\cdot(\mathbf{N}\cdot\mathbf{E}) = (\mathbf{E}\cdot\mathbf{N})\cdot(\mathbf{N}\cdot\mathbf{E}\cdot\mathbf{N})^{-1}\cdot(\mathbf{N}\cdot\mathbf{E}). \tag{46}$$

The tensor $\mathbf{E}^N$ is fully symmetric since the elastic acoustic tensor is symmetric and satisfies identically the relation $\mathbf{N}\cdot\mathbf{E}^N\cdot\mathbf{N} = \mathbf{N}\cdot\mathbf{E}\cdot\mathbf{N} = \mathbf{Q}$. Since the inverse of the elastic acoustic tensor is positive definite, the tensor $\mathbf{E}^N$ is positive semidefinite: the associated quadratic form $(\mathbf{x}:\mathbf{E}\cdot\mathbf{N})\cdot\mathbf{Q}^{-1}\cdot(\mathbf{N}\cdot\mathbf{E}\cdot\mathbf{x}) \geqslant 0$, for any tensor $\mathbf{x}$, is positive, or zero if the traction vectors in the parentheses vanish. The localization indicator (44) may be written as

$$q_s = 1 - \left(\frac{\mathbf{n}:\mathbf{E}^N:\mathbf{m}}{\bar{H}}\right) - \left(\frac{(\mathbf{n}:\mathbf{E}^N:\mathbf{n})(\mathbf{m}:\mathbf{E}^N:\mathbf{m}) - (\mathbf{n}:\mathbf{E}^N:\mathbf{m})^2}{4\bar{H}^2}\right) \tag{47}$$

where only the second parenthesis is zero in the case of the weak indicator q. Equation (47) compares directly with the previous results for the continuous failure indicator (37) and may be visualized by plots entirely analogous to those in Fig. 1. The critical hardening parameters for localization become:

$$\begin{aligned} H_{df}^{N} &= \mathbf{n}:(\mathbf{E}^N - \mathbf{E}):\mathbf{m} \\ H_{sdf}^{N} &= H_{df}^{N} + \tfrac{1}{2}(\sqrt{(\mathbf{n}:\mathbf{E}^N:\mathbf{n})(\mathbf{m}:\mathbf{E}^N:\mathbf{m})} - \mathbf{n}:\mathbf{E}^N:\mathbf{m}) \geqslant H_{df}^{N}. \end{aligned} \tag{48}$$

These expressions are analogous to the conditions for continuous failure (34) and (39). Note that the bilinear form $\mathbf{n}:(\mathbf{E}^N - \mathbf{E}):\mathbf{m}$ and, hence, H_{df}^{N} can be positive, negative or zero. For associative flow $H_{df}^{N} = H_{sdf}^{N}$ must be negative. In fact, since the loss of stability for $H_{cf} = H_{scf} = 0$ is a necessary condition for any kind of bifurcation, it follows that $\mathbf{n}:\mathbf{E}^N:\mathbf{n} \leqslant \mathbf{n}:\mathbf{E}:\mathbf{n}$.

The onset of weak localization is obtained by maximizing the hardening parameters H_{df}^{N} with respect to $\mathbf{N}$ for $\|\mathbf{N}\| = 1$, in order to obtain the critical localization direction H_{df} and the critical hardening parameter H_{df} for discontinuous failure. Equivalently, one might minimize q with respect to $\mathbf{N}$ at fixed $\bar{H}$, and then set $\tilde{q} = 0$ [see eqn (25)], to obtain the same result.

If the elastic stiffness has the isotropic format $\mathbf{E} = 2\mu\mathbf{i}_4^s + \lambda\mathbf{i}_2\otimes\mathbf{i}_2$, then the elastic localization tensor and its inverse may be derived by the Shermann and Morrison (1950) formula:

$$\mathbf{Q} = \mu\mathbf{i}_2 + (\lambda+\mu)\mathbf{N}\otimes\mathbf{N} \;\Rightarrow\; \mathbf{Q}^{-1} = \frac{1}{\mu}\mathbf{i}_2 - \frac{\lambda+\mu}{\mu(\lambda+2\mu)}\mathbf{N}\otimes\mathbf{N}. \tag{49}$$

In the isotropic case the projection of the elastic stiffness $\mathbf{E}^N$ simplifies, so that the product $\mathbf{n}:\mathbf{E}^N:\mathbf{m}$ may be evaluated in analytical form. In fact, introducing the notation $o_N = \mathbf{N}\cdot\mathbf{o}\cdot\mathbf{N}$

(which denotes the component of **o** in the direction **N**), the localization condition (42) turns out to be:

$$\bar{H}^{\mathrm{N}}_{\mathrm{df}} = \mathbf{n}:\mathbf{E}^{\mathrm{N}}:\mathbf{m} = \frac{\lambda^2}{\lambda+2\mu}\,\mathrm{tr}\,[\mathbf{n}]\mathrm{tr}\,[\mathbf{m}] + \frac{2\lambda\mu}{\lambda+2\mu}(\mathrm{tr}\,[\mathbf{n}]m_{\mathrm{N}} + \mathrm{tr}\,[\mathbf{m}]n_{\mathrm{N}}) + 4\mu(\mathbf{n}\cdot\mathbf{m})_{\mathrm{N}} - 4\mu\frac{(\lambda+\mu)}{\lambda+2\mu}n_{\mathrm{N}}m_{\mathrm{N}}. \tag{50}$$

Using the representation theorems of isotropic tensor functions the gradients **n** and **m** may be expanded up to second order terms in stress (or strain): e.g. for isotropic loading conditions, $\mathbf{n} = f_1[\boldsymbol{\sigma}]\mathbf{i}_2 + f_2[\boldsymbol{\sigma}]\boldsymbol{\sigma} + f_3[\boldsymbol{\sigma}]\boldsymbol{\sigma}^2$ and analogously for **m**. Such expansions permit a geometric representation of localization by means of a fourth order curve in the Mohr stress (or strain) space coordinates (Benallal and Comi, 1996; Rizzi *et al.*, 1995). The stationarity condition to be used in the aforementioned maximization of the hardening modulus $H^{\mathrm{N}}_{\mathrm{df}}$ with respect to **N** reduces to

$$\frac{\partial(\mathbf{n}:\mathbf{E}^{\mathrm{N}}:\mathbf{m})}{\partial\mathbf{N}} = \mathbf{0}. \tag{51}$$

Considering that $\partial o_{\mathrm{N}}/\partial\mathbf{N} = 2\mathbf{o}\cdot\mathbf{N}$ for the symmetric second order tensor **o** and by differentiating eqn (50) with respect to **N**, eqn (51) results in the following homogeneous system of equations which needs to be solved for the critical localization directions:

$$(\lambda(\mathrm{tr}[\mathbf{n}]\mathbf{m} + \mathrm{tr}[\mathbf{m}]\mathbf{n}) + 2(\lambda+2\mu)(\mathbf{n}\cdot\mathbf{m}) - 2(\lambda+\mu)(m_{\mathrm{N}}\mathbf{n} + n_{\mathrm{N}}\mathbf{m}))\cdot\mathbf{N} = \mathbf{0}. \tag{52}$$

In order to obtain the full eigenspectrum of $\mathbf{Q}_t$ in isotropic elasticity, the tangent localization tensor is the result of two rank-one updates of a tensor proportional to $\mathbf{i}_2$:

$$\mathbf{Q}_t = \mu\mathbf{i}_2 + (\lambda+\mu)\mathbf{N}\otimes\mathbf{N} - \frac{\mathbf{a}\otimes\mathbf{b}}{\bar{H}} = \mu\mathbf{i}_2 + \mathbf{U}_2. \tag{53}$$

Then, according to Appendix A, the eigensolution for $\mathbf{U}_2$ may be obtained by assuming eigenvectors of the form $\mathbf{M} = \alpha_1\mathbf{N} + \alpha_2\mathbf{a}$, which leads to the linear system of two simultaneous equations, cf. eqn (A.6):

$$\begin{cases} (\lambda+\mu-u)\quad \alpha_1 + \quad (\lambda+\mu)\mathbf{N}\cdot\mathbf{a} \quad \alpha_2 = 0 \\ \dfrac{-\mathbf{N}\cdot\mathbf{b}}{\bar{H}} \quad \alpha_1 + \quad \left(\dfrac{-\mathbf{a}\cdot\mathbf{b}}{\bar{H}} - u\right) \quad \alpha_2 = 0 \end{cases} \tag{54}$$

and to the closed-form expression for the eigenvalues, cf. eqn (A.10):

$$u_{1,2} = \frac{1}{2}\left(\left((\lambda+\mu) - \frac{\mathbf{a}\cdot\mathbf{b}}{\bar{H}}\right) \pm \sqrt{\left((\lambda+\mu) + \frac{\mathbf{a}\cdot\mathbf{b}}{\bar{H}}\right)^2 - 4\frac{(\lambda+\mu)}{\bar{H}}\mathbf{N}\cdot\mathbf{a}\mathbf{N}\cdot\mathbf{b}}\right). \tag{55}$$

The eigenvalues of the localization tensor $\mathbf{Q}_t$ are then $\omega_{1,2} = \mu + u_{1,2}$ and $\omega_3 = \mu$. This means that one elastic shear wave velocity is not modified, while the other one and the elastic longitudinal wave velocity are changed in accordance with the anisotropy induced by the inelastic process. Note that the eigenvalues of the elastic acoustic tensor are recovered for $\bar{H} \to \infty$ as $\omega_1 = \lambda + 2\mu$, $\omega_2 = \mu$ and $\omega_3 = \mu$, which are proportional to the wave velocities for an elastic isotropic medium ($\omega_i = \rho c_i^2$).

The localization condition $\det[\mathbf{Q}_t] = 0$ is fulfilled when the minimum eigenvalue ω_2 vanishes, i.e. for $u_2 = -\mu$. This condition combined with eqn (55) provides the equation for the critical hardening parameter necessary for localization at a given direction $\mathbf{N}$:

$$\bar{H}^{\mathrm{N}}_{\mathrm{df}} = \frac{(\lambda+2\mu)\mathbf{a}\cdot\mathbf{b}-(\lambda+\mu)\mathbf{N}\cdot\mathbf{a}\mathbf{N}\cdot\mathbf{b}}{\mu(\lambda+2\mu)}. \tag{56}$$

This result could also be obtained by substituting expression (49) for the isotropic $\mathbf{Q}^{-1}$ into eqn (42). Substituting eqn (49) in eqn (41), the eigenvalues are expressed in the form obtained by Ottosen and Runesson (1991b) [see Rizzi, 1995], which allows a rational discussion on the occurrence of negative or complex eigenvalues.

The eigenvectors $\mathbf{M}^{1,2} = \alpha_1^{1,2}\mathbf{N}+\alpha_2^{1,2}\mathbf{a}$ of the localization tensor corresponding to $\omega_{1,2}$ are the solution of the system of equations in eqn (54) once the eigenvalues are found. From the first equation we obtain

$$\frac{\alpha_1^{1,2}}{\alpha_2^{1,2}} = \frac{\mathbf{N}\cdot\mathbf{a}(\lambda+\mu)}{\omega_{1,2}-(\lambda+2\mu)} \tag{57}$$

which coincides with the result developed earlier on by Ottosen and Runesson (1991b). At the onset of localization, when $\omega_2 = 0$, the eigenvector $\mathbf{M}^2$ characterizes the mode shape of discontinuous bifurcation:

$$\mathbf{M}^2 = -\mathbf{N}\cdot\mathbf{a}\frac{(\lambda+\mu)}{(\lambda+2\mu)}\mathbf{N}+\mathbf{a} \tag{58}$$

and in general represents a mixed mode condition of failure. Mode I failure with $\mathbf{M}^2\|\mathbf{N}$ is recovered when $\mathbf{a}\|\mathbf{N}$, while mode II failure with $\mathbf{M}^2\perp\mathbf{N}$ occurs when $\mathbf{a}\perp\mathbf{N}$.

From the definition of the traction vector $\mathbf{a}$, and from the isotropic format of $\mathbf{E}$, it follows that:

$$\mathbf{a} = \mathbf{N}\cdot\mathbf{E}:\mathbf{m} = \lambda\mathrm{tr}[\mathbf{m}]\mathbf{N}+2\mu\mathbf{m}\cdot\mathbf{N}, \quad \mathbf{N}\cdot\mathbf{a} = \lambda\,\mathrm{tr}\,[\mathbf{m}]+2\mu\mathbf{N}\cdot\mathbf{m}\cdot\mathbf{N}. \tag{59}$$

If $\mathbf{N}$ is aligned with one of the principal directions of $\mathbf{m}$ and also if $\mathbf{m}$ is purely volumetric, then the localization mode is purely decohesive (mode I). If $\mathbf{m}$ is purely deviatoric, mode II failure is only achieved when the direction $\mathbf{N}$ is perpendicular to the shearing direction, such that $\mathbf{N}\cdot\mathbf{m}\cdot\mathbf{N} = 0$.

5. SPECTRAL ANALYSIS OF TANGENT OPERATORS FOR DOUBLE-DISSIPATION MODELS

As a meaningful case of a double-dissipation model ($n = 2$) we consider a combination of plasticity and elastic degradation. The two inelastic mechanisms give rise to the decomposition of the infinitesimal strain rate into $\dot{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{\varepsilon}}_{\mathrm{E}}+\dot{\boldsymbol{\varepsilon}}_{\mathrm{I}}$, where $\dot{\boldsymbol{\varepsilon}}_{\mathrm{I}} = \dot{\boldsymbol{\varepsilon}}_{\mathrm{D}}+\dot{\boldsymbol{\varepsilon}}_{\mathrm{P}} = \dot{\lambda}_{\mathrm{D}}\mathbf{m}_{\mathrm{D}}+\dot{\lambda}_{\mathrm{P}}\mathbf{m}_{\mathrm{P}}$. Here $\dot{\lambda}_{\mathrm{D}}$, $\dot{\lambda}_{\mathrm{P}}$ denote the damage and plastic multipliers which measure the magnitude of the inelastic strain contributions and $\mathbf{m}_{\mathrm{D}}$, $\mathbf{m}_{\mathrm{P}}$ represent the tensor-valued directions of elastic degradation and plastic flow. While $\mathbf{m}_{\mathrm{P}}$ is normally expressed as the gradient of a plastic potential, $\mathbf{m}_{\mathrm{D}}$ derives from a fourth order tensor evolution law of the elastic compliance $\dot{\mathbf{C}} = \dot{\lambda}_{\mathrm{D}}\boldsymbol{M}$, which assigns the direction of elastic degradation in terms of $\mathbf{m}_{\mathrm{D}} = \boldsymbol{M}:\boldsymbol{\sigma}$ (Carol *et al.*, 1994).

Two strict inequalities, $F_{\mathrm{D}}[\boldsymbol{\sigma}, \lambda_{\mathrm{D}}, \lambda_{\mathrm{P}}] < 0$ and $F_{\mathrm{P}}[\boldsymbol{\sigma}, \lambda_{\mathrm{D}}, \lambda_{\mathrm{P}}] < 0$, define all stress states in which no further dissipation takes place. Their gradients and the general format of the (2×2) hardening matrix $\mathbf{H} = [-\partial F_\alpha/\partial\lambda_\beta]$ and the matrix of critical softening $\mathbf{H}_{\mathrm{c}} = [-\mathbf{n}_\alpha:\mathbf{E}:\mathbf{m}_\beta]$ with ${}_{\alpha,\beta} = {}_{\mathrm{D,P}}$ are

$$\mathbf{n}_{\mathrm{D}} = \frac{\partial F_{\mathrm{D}}}{\partial \boldsymbol{\sigma}}; \quad \mathbf{n}_{\mathrm{P}} = \frac{\partial F_{\mathrm{P}}}{\partial \boldsymbol{\sigma}}; \quad \mathbf{H} = \begin{bmatrix} H_{\mathrm{DD}} & H_{\mathrm{DP}} \\ H_{\mathrm{PD}} & H_{\mathrm{PP}} \end{bmatrix}; \quad \mathbf{H}_{\mathrm{c}} = \begin{bmatrix} -\mathbf{n}_{\mathrm{D}} : \mathbf{E} : \mathbf{m}_{\mathrm{D}} & -\mathbf{n}_{\mathrm{D}} : \mathbf{E} : \mathbf{m}_{\mathrm{P}} \\ -\mathbf{n}_{\mathrm{P}} : \mathbf{E} : \mathbf{m}_{\mathrm{D}} & -\mathbf{n}_{\mathrm{P}} : \mathbf{E} : \mathbf{m}_{\mathrm{P}} \end{bmatrix}. \tag{60}$$

5.1. *Singularity analysis of the tangent stiffness tensor*

When the two inelastic processes are active, the solution of the consistency equations $\dot{F}_{\mathrm{D}} = 0$ and $\dot{F}_{\mathrm{P}} = 0$ requires inversion of the (2×2) effective hardening matrix $\bar{\mathbf{H}} = \mathbf{H} - \mathbf{H}_{\mathrm{c}}$ which is assumed to be a P-matrix. From the two consistency conditions, the inelastic multipliers are expressed in terms of the imposed strain rates:

$$\dot{\lambda}_{\mathrm{D}} = \frac{(\bar{H}_{\mathrm{PP}}\bar{\mathbf{n}}_{\mathrm{L}} - \bar{H}_{\mathrm{DP}}\bar{\mathbf{n}}_{\mathrm{P}})}{\det[\bar{\mathbf{H}}]} : \dot{\boldsymbol{\varepsilon}} = \hat{\mathbf{n}}_{\mathrm{D}} : \dot{\boldsymbol{\varepsilon}}; \quad \dot{\lambda}_{\mathrm{P}} = \frac{(-\bar{H}_{\mathrm{PD}}\bar{\mathbf{n}}_{\mathrm{D}} + \bar{H}_{\mathrm{DD}}\bar{\mathbf{n}}_{\mathrm{P}})}{\det[\bar{\mathbf{H}}]} : \dot{\boldsymbol{\varepsilon}} = \hat{\mathbf{n}}_{\mathrm{P}} : \dot{\boldsymbol{\varepsilon}} \tag{61}$$

where $\bar{\mathbf{n}}_{\alpha} = \mathbf{E} : \mathbf{n}_{\alpha}$ and $\hat{\mathbf{n}}_{\alpha} = \bar{\mathbf{H}}_{\alpha\beta}^{-1}\bar{\mathbf{n}}_{\beta}$.

Consequently, the tangent stiffness operator (12) results in two rank-one updates of the current elastic stiffness tensor:

$$\mathbf{E}_{\mathrm{t}} = \mathbf{E} + \bar{\mathbf{m}}_{\mathrm{D}} \otimes \hat{\mathbf{n}}_{\mathrm{D}} + \bar{\mathbf{m}}_{\mathrm{P}} \otimes \hat{\mathbf{n}}_{\mathrm{P}} \tag{62}$$

where $\bar{\mathbf{m}}_{\alpha} = -\mathbf{E} : \mathbf{m}_{\alpha}$. The tangent operator is in general non-symmetric. It turns symmetric if both dissipation rules are associative ($\mathbf{m}_{\alpha} = \mathbf{n}_{\alpha}$ or $\bar{\mathbf{m}}_{\alpha} = -\bar{\mathbf{n}}_{\alpha}$) and if $\bar{\mathbf{H}}$ is symmetric. If $\bar{\mathbf{H}}$ is diagonal, the two dissipation processes decouple, and each dissipation multiplier may be computed independently from each other. Clearly, this occurs when $H_{\mathrm{DP}} = H^{\mathrm{c}}_{\mathrm{DP}}$, $\mathrm{H}_{\mathrm{PD}} = \mathrm{H}^{\mathrm{c}}_{\mathrm{PD}}$, which means that the hardening mechanisms are coupled.

The limit-point condition, i.e. the weak failure condition associated with the singularity of the tangent stiffness operator (62), may be formulated by using the expression (A.12) for the determinant of two rank-one updates of the identity matrix. From eqn (62) we can write:

$$\mathbf{E}^{-1} : \mathbf{E}_{\mathrm{t}} = \mathbf{i}_4^{\mathrm{s}} - \mathbf{m}_{\mathrm{D}} \otimes \hat{\mathbf{n}}_{\mathrm{D}} - \mathbf{m}_{\mathrm{P}} \otimes \hat{\mathbf{n}}_{\mathrm{P}} \tag{63}$$

where $\hat{\mathbf{n}}_{\mathrm{D}} = (\bar{H}_{\mathrm{PP}}\bar{\mathbf{n}}_{\mathrm{D}} - \bar{H}_{\mathrm{DP}}\bar{\mathbf{n}}_{\mathrm{P}})/\det[\bar{\mathbf{H}}]$ and $\hat{\mathbf{n}}_{\mathrm{P}} = (-\bar{H}_{\mathrm{PD}}\bar{\mathbf{n}}_{\mathrm{D}} + \bar{H}_{\mathrm{DD}}\bar{\mathbf{n}}_{\mathrm{P}})/\det[\bar{\mathbf{H}}]$. When the perturbation formula (A.12) is applied to eqn (63), the limit-point condition is of the form:

$$e = \frac{\det[\mathbf{E}_{\mathrm{t}}]}{\det[\mathbf{E}]} = (1 - \mathbf{m}_{\mathrm{D}} : \hat{\mathbf{n}}_{\mathrm{D}})(1 - \mathbf{m}_{\mathrm{P}} : \hat{\mathbf{n}}_{\mathrm{P}}) - (\mathbf{m}_{\mathrm{D}} : \hat{\mathbf{n}}_{\mathrm{P}})(\mathbf{m}_{\mathrm{P}} : \hat{\mathbf{n}}_{\mathrm{D}}) = \frac{\det[\mathbf{H}]}{\det[\bar{\mathbf{H}}]} = 0. \tag{64}$$

This condition may be interpreted as a constraint on the hardening parameters H_{DD}, H_{PP}, H_{DP} and H_{PD} of the hardening matrix $\mathbf{H}_{\mathrm{cf}}$ which marks the onset of continuous failure, $e[\mathbf{H}_{\mathrm{cf}}] = 0$, when $\det[\mathbf{H}] = 0$.

Remarks. (a) Let the effective hardening matrix $\bar{\mathbf{H}}$ be symmetric and hence positive definite since it has been assumed to be a P-matrix. If the critical hardening matrix $\mathbf{H}_{\mathrm{c}}$ defined by eqn (60) was symmetric and positive definite, then we could write (Mirsky, 1955):

$$\det[\mathbf{H}] = \det[\bar{\mathbf{H}} + \mathbf{H}_{\mathrm{c}}] > \det[\bar{\mathbf{H}}] + \det[\mathbf{H}_{\mathrm{c}}] > 0 \;\Rightarrow\; \frac{\det[\mathbf{H}]}{\det[\bar{\mathbf{H}}]} > 1. \tag{65}$$

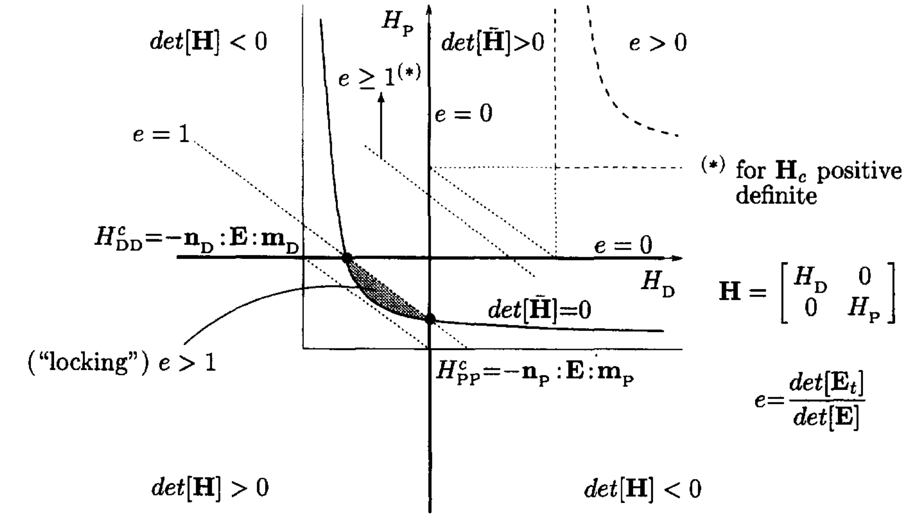

Fig. 3. Continuous failure indicator e for dissipation mechanisms independent in the stress space.

In this case (Figs 3 and 4) the failure indicator e would be positive and larger than 1, giving rise to locking of the tangent stiffness tensor in terms of the global determinant measure. Note that in this case $-\mathbf{n}_{(\alpha)}:\mathbf{E}:\mathbf{m}_{(\alpha)} > 0$ for the diagonal elements of $\mathbf{H}_c$ and, that for associative laws $\mathbf{H}_c = -\boldsymbol{n}:\mathbf{E}:\boldsymbol{n}$ is negative definite, i.e. $-\mathbf{n}_{(\alpha)}:\mathbf{E}:\mathbf{n}_{(\alpha)} < 0$ and $\det[\mathbf{H}_c] > 0$. The positiveness of $\det[\mathbf{H}_c]$ agrees with the considerations in Section 4 (Fig. 2) which demonstrate that $\mathbf{n}_{(\alpha)}:\mathbf{E}:\mathbf{n}_{(\alpha)} > |\mathbf{n}_{(\alpha)}:\mathbf{E}:\mathbf{n}_{(\beta)}|$. These remarks suggest that the matrix $\mathbf{H}_c$ is negative definite, even though this does not preclude locking in general (Figs 3–5). On the contrary, for single-dissipative models locking is ruled out when $H_c < 0$ (cf. Section 4).

(b) In the case of independent dissipation mechanisms in stress space, when $\mathbf{H}$ is diagonal (like for Koiter's non-interacting hardening), the limit-point condition is satisfied for $H_D = 0$ or $H_P = 0$, i.e. along the two reference axes in the (H_D, H_P) plane of Fig. 3. The

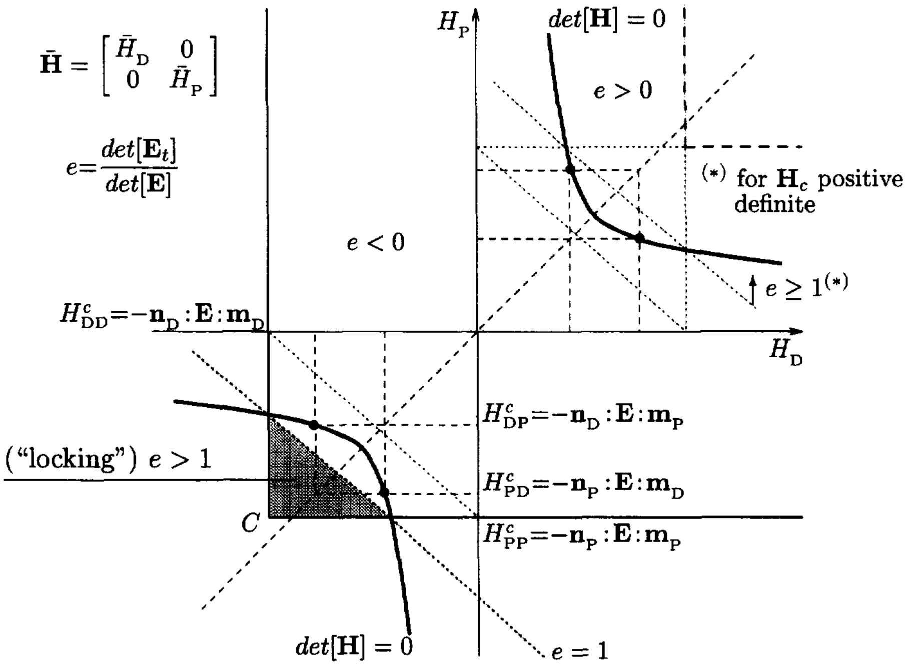

Fig. 4. Continuous failure indicator e for dissipation mechanisms independent in the strain space.

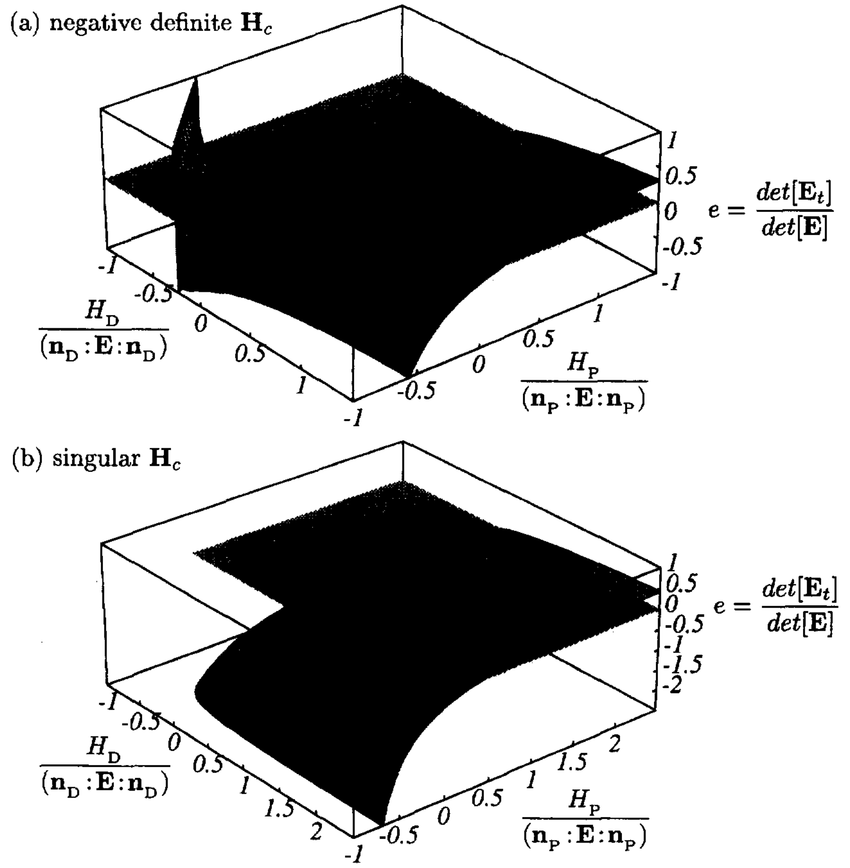

Fig. 5. Continuous failure indicator e in the associative, fully uncoupled case: (a) for negative definite critical hardening matrix $\mathbf{H}_c$; (b) for singular $\mathbf{H}_c$.

two hardening parameters make $\det[\bar{\mathbf{H}}]$ positive, in the region above the hyperbola, the equation of which represents the singularity condition for matrix $\bar{\mathbf{H}}$:

$$\det[\bar{\mathbf{H}}] = H_D H_P + (\mathbf{n}_P : \mathbf{E} : \mathbf{m}_P) H_D + (\mathbf{n}_D : \mathbf{E} : \mathbf{m}_D) H_P + \det[\mathbf{H}_c] = 0. \tag{66}$$

The region where $e \geqslant 1$ according to eqn (64) is the half-plane in Fig. 3:

$$(\mathbf{n}_P : \mathbf{E} : \mathbf{m}_P) H_D + (\mathbf{n}_D : \mathbf{E} : \mathbf{m}_D) H_P + \det[\mathbf{H}_c] \leqslant 0. \tag{67}$$

In the intersection of the two regions, i.e. in the shaded area of the third quadrant, locking is possible even for a negative definite $\mathbf{H}_c$. If hardening is still positive for one mechanism and zero for the other, uniqueness of $\dot{\boldsymbol{\varepsilon}}$ for a given $\dot{\boldsymbol{\sigma}}$ is lost (this would not be the case if only the first mechanism would be active). When both are active, interaction of the two dissipation mechanisms destabilizes the tangent operator and leads to possible locking in the softening/softening range.

(c) In the fully uncoupled case with diagonal matrix $\bar{\mathbf{H}}$, i.e. for independent mechanisms in the strain space, eqn (64) simplifies to

$$e = \left(1 - \frac{\mathbf{m}_D : \mathbf{E} : \mathbf{n}_D}{\bar{H}_D}\right)\left(1 - \frac{\mathbf{m}_P : \mathbf{E} : \mathbf{n}_P}{\bar{H}_P}\right) - \left(\frac{\mathbf{m}_D : \mathbf{E} : \mathbf{n}_P}{\bar{H}_P}\right)\left(\frac{\mathbf{m}_P : \mathbf{E} : \mathbf{n}_D}{\bar{H}_D}\right)$$

$$= \frac{H_D H_P - (\mathbf{m}_D : \mathbf{E} : \mathbf{n}_P)(\mathbf{m}_P : \mathbf{E} : \mathbf{n}_D)}{(H_D + \mathbf{m}_D : \mathbf{E} : \mathbf{n}_D)(H_P + \mathbf{m}_P : \mathbf{E} : \mathbf{n}_P)} = 0. \tag{68}$$

If matrix $\mathbf{H}_c$ was positive definite, the limit-point condition could never be reached. If $\mathbf{H}_c$ is negative definite, which implies $(\mathbf{n}_D : \mathbf{E} : \mathbf{m}_D)\,(\mathbf{n}_P : \mathbf{E} : \mathbf{m}_P) > (\mathbf{n}_D : \mathbf{E} : \mathbf{m}_P)\,(\mathbf{n}_P : \mathbf{E} : \mathbf{m}_D)$, we obtain the situation depicted in Fig. 4 where it is assumed that H^c_{DP} and H^c_{PD} have the same (negative) sign. The half plane $e \geqslant 1$ is still defined by eqn (67) with $\det[\mathbf{H}_c]$ replaced by $H^c_{DD}H^c_{PP} + H^c_{DP}H^c_{PD}$. The shaded area in Fig. 4 indicates locking even with $\mathbf{H}_c$ being negative definite. The limit point can be attained with positive hardening parameters belonging to the hyperbola $H_D H_P = H^c_{DP}H^c_{PD}$. Thus, the interaction of plasticity and damage can activate the limit-point condition before one of the two active dissipation mechanisms reaches the limit point. In the softening/softening region the determinant of the tangent stiffness can reach positive values and may even lead to locking. The same remarks apply to the associative case depicted in Fig. 5a.

(d) Locking of the tangent stiffness can be avoided if matrix $\mathbf{H}_c$ is indefinite, with $\det[\mathbf{H}_c] \leqslant 0$, while the diagonal elements are still negative. In this case point C belongs to or lies above the lower hyperbola in Fig. 4, and e remains negative throughout the admissible range $H_D > H^c_D, H_P > H^c_P$. In fact, eqn (68) may be written

$$e = 1 - \frac{\mathbf{n}_D : \mathbf{E} : \mathbf{m}_D}{\bar{H}_D} - \frac{\mathbf{n}_P : \mathbf{E} : \mathbf{m}_P}{\bar{H}_P} - \frac{-\det[\mathbf{H}_c]}{\bar{H}_D \bar{H}_P} \tag{69}$$

which shows that $e < 1$ for $\mathbf{n}_{(\alpha)} : \mathbf{E} : \mathbf{m}_{(\alpha)} > 0$ and $\det[\mathbf{H}_c] \leqslant 0$. In the associative case, when $\det[\mathbf{H}_c] \geqslant 0$ (the equality holds only for linearly dependent tensors), locking is eliminated only for singular $\mathbf{H}_c$, which is the case when $\mathbf{n}_D \| \mathbf{n}_P$. In this case e decreases continuously toward $-\infty$ as shown in Fig. 5b.

5.2. *Singularity analysis of the tangent localization tensor*

The weak localization condition requires a singularity of the tangent localization tensor comprised of two rank-one updates of the elastic acoustic tensor. This condition is dealt with below by applying the perturbation formula (A.12) to the tensor $\mathbf{Q}^{-1} \cdot \mathbf{Q}_t$:

$$\mathbf{Q}_t = \mathbf{Q} - \mathbf{a}_D \otimes \mathbf{d}_D - \mathbf{a}_P \otimes \mathbf{d}_P \;\Rightarrow\; \mathbf{Q}^{-1} \cdot \mathbf{Q}_t = \mathbf{i}_2 - \mathbf{Q}^{-1} \cdot \mathbf{a}_D \otimes \mathbf{d}_D - \mathbf{Q}^{-1} \cdot \mathbf{a}_P \otimes \mathbf{d}_P \tag{70}$$

where $\mathbf{a}_\alpha = \mathbf{N} \cdot \mathbf{E} : \mathbf{m}_\alpha$ and $\mathbf{d}_\alpha = \bar{H}^{-1}_{\alpha\beta}\mathbf{n}_\beta : \mathbf{E} \cdot \mathbf{N}$ are the traction vectors with $_\alpha = {}_{D,P}$. Thus, the non-dimensional localization indicator q has the form:

$$q = \frac{\det[\mathbf{Q}_t]}{\det[\mathbf{Q}]} = (1 - \mathbf{d}_D \cdot \mathbf{Q}^{-1} \cdot \mathbf{a}_D)(1 - \mathbf{d}_P \cdot \mathbf{Q}^{-1} \cdot \mathbf{a}_P) - (\mathbf{d}_D \cdot \mathbf{Q}^{-1} \cdot \mathbf{a}_P)(\mathbf{d}_P \cdot \mathbf{Q}^{-1} \cdot \mathbf{a}_D). \tag{71}$$

Each term $\mathbf{d}_\alpha \cdot \mathbf{Q}^{-1}\mathbf{a}_\beta$ may be written in explicit form by defining the matrix $\mathbf{H}^N_c$ as $(\mathbf{H}^N_c)_{\alpha\beta} = -\mathbf{n}_\alpha : \mathbf{E}^N : \mathbf{m}_\beta$, in analogy to the critical softening matrix and the non-dimensional hardening matrix $\mathbf{h}^N_c$ as $(\mathbf{h}^N_c)_{\alpha\beta} = \bar{H}^{-1}_{\alpha\gamma}\mathbf{n}_\gamma : \mathbf{E}^N : \mathbf{m}_\beta$, where $\mathbf{E}^N$ denotes the fourth order stiffness projection defined in eqn (46). Thus, the localization indicator q is similar to e in the first row of eqn (64), which demonstrates its dependence on the properties of the matrix $\mathbf{h}^N_c$:

$$q = 1 - \mathbf{h}^N_{cDD} - \mathbf{h}^N_{cPP} - (-\det[\mathbf{h}^N_c]). \tag{72}$$

The occurrence of $q > 1$, which we may interpret as an acceleration effect of inelastic waves with respect to the elastic wave speed, is possible, even though it is hardly meaningful if reached after the localization condition $q = 0$ is satisfied.

In the fully decoupled case $\bar{\mathbf{H}}$ is diagonal and it is possible to rewrite eqns (71) and (72) in analogy to e in eqns (68) and (69) [$\mathbf{E}$ corresponding to $\mathbf{E}^N$ and $\mathbf{H}_c$ to $\mathbf{H}^N_c$]:

$$q = 1 - \frac{\mathbf{n}_D : \mathbf{E}^N : \mathbf{n}_D}{\bar{H}_D} - \frac{\mathbf{n}_P : \mathbf{E}^N : \mathbf{n}_P}{\bar{H}_P} - \frac{-\det[\mathbf{H}^N_c]}{\bar{H}_D \bar{H}_P}. \tag{73}$$

Comments similar to those drawn from Figs 4 and 5 hold qualitatively also in this case. In fact, the properties of the hardening matrix $\mathbf{H}_c^N$ govern the behavior of q: when $\mathbf{H}_c^N$ is positive definite, localization cannot occur as $q \geqslant 1$; when $\mathbf{H}_c^N$ is negative definite, singularity may arise, but the acceleration effect cannot be ruled out *a priori*, unless by prescribing appropriate lower bounds for softening; if $\mathbf{H}_c^N$ is indefinite, with $\det[\mathbf{H}_c^N] \leqslant 0$ and negative diagonal elements, the spurious acceleration effect is removed.

6. APPLICATION TO VON MISES PLASTICITY COMBINED WITH SCALAR DAMAGE

As an illustration of the results above, we consider a particular double-dissipative constitutive model which combines plasticity with elastic degradation. The J_2-plastic dissipation mechanism follows the von Mises yield function with associative flow rule. The elastic degradation is a scalar-valued associative formulation for isotropic damage (Carol *et al.*, 1994): all elastic stiffness and compliance moduli evolve according to the scalar damage law $\mathbf{E} = (1-D)\mathbf{E}_0$, $\mathbf{C} = \mathbf{C}_0/(1-D)$, where D denotes the scalar damage variable, $0 \leqslant D < 1$, which is related to the damage multiplier λ_D.

The two loading conditions for the coupled stress-based formulation read:

$$F_D = \tfrac{1}{2}\boldsymbol{\sigma}:\mathbf{C}:\boldsymbol{\sigma} - s_D[\lambda_D, \lambda_P] = 0; \quad F_P = \tfrac{1}{2}\boldsymbol{\sigma}:\mathbf{C}_d:\boldsymbol{\sigma} - s_P[\lambda_D, \lambda_P] = 0. \tag{74}$$

Here, s_D and s_P define the damage and plastic thresholds, and $\mathbf{C}_d$ represents the deviatoric part of the current (damaged) elastic compliance tensor according to the spectral decomposition:

$$\mathbf{C} = \mathbf{C}_v + \mathbf{C}_d = \frac{1}{3K}\mathbf{P}_v + \frac{1}{2G}\mathbf{P}_d; \quad \mathbf{P}_v = \tfrac{1}{3}\mathbf{i}_2 \otimes \mathbf{i}_2; \quad \mathbf{P}_d = \mathbf{i}_4^s - \mathbf{P}_v \tag{75}$$

where K denotes the bulk modulus and G the shear modulus. The projection operators $\mathbf{P}_v$ and $\mathbf{P}_d$ extract the volumetric and deviatoric components of a symmetric second order tensor (Neilsen and Schreyer, 1993). In the present case, setting $C_v = (3K)^{-1}$ and $C_d = (2G)^{-1}$, we have:

$$\mathbf{C}_d:\boldsymbol{\sigma} = C_d\boldsymbol{\sigma}_d \quad \text{and} \quad \boldsymbol{\sigma}:\mathbf{C}_d:\boldsymbol{\sigma} = C_d\boldsymbol{\sigma}_d:\boldsymbol{\sigma}_d. \tag{76}$$

In the associative case the gradients of the loading surfaces define the directions of the inelastic strain rates:

$$\mathbf{m}_D = \mathbf{n}_D = \mathbf{C}:\boldsymbol{\sigma} = C_v\boldsymbol{\sigma}_v + C_d\boldsymbol{\sigma}_d; \quad \mathbf{m}_P = \mathbf{n}_P = \mathbf{C}_d:\boldsymbol{\sigma} = C_d\boldsymbol{\sigma}_d. \tag{77}$$

The matrix of critical softening reduces to

$$\mathbf{H}_c = \begin{bmatrix} -\mathbf{n}_D:\mathbf{E}:\mathbf{n}_D & -\mathbf{n}_D:\mathbf{E}:\mathbf{n}_P \\ -\mathbf{n}_P:\mathbf{E}:\mathbf{n}_D & -\mathbf{n}_P:\mathbf{E}:\mathbf{n}_P \end{bmatrix} = \begin{bmatrix} -(C_v\boldsymbol{\sigma}_v:\boldsymbol{\sigma}_v + C_d\boldsymbol{\sigma}_d:\boldsymbol{\sigma}_d) & -C_d\boldsymbol{\sigma}_d:\boldsymbol{\sigma}_d \\ -C_d\boldsymbol{\sigma}_d:\boldsymbol{\sigma}_d & -C_d\boldsymbol{\sigma}_d:\boldsymbol{\sigma}_d \end{bmatrix}. \tag{78}$$

Matrix $\mathbf{H}_c$ is negative semidefinite since its diagonal elements are negative or zero, whereas its determinant $\det[\mathbf{H}_c] = (C_v\boldsymbol{\sigma}_v:\boldsymbol{\sigma}_v)\ (C_d\boldsymbol{\sigma}_d:\boldsymbol{\sigma}_d) \geqslant 0$ is positive or zero. The determinant vanishes only if the stress tensor is either purely volumetric or purely deviatoric, or in the case of incompressible elasticity (i.e. $\nu = 0.5$ so that $C_v = 0$). According to the remarks of Section 4, locking of the tangent response is precluded for all these cases.

The analysis of the localization indicator may be detailed for the case of diagonal hardening $\bar{\mathbf{H}} = \mathbf{H} - \mathbf{H}_c$, which occurs when the functions s_D and s_P in eqn (74) are chosen such that the hardening matrix $\mathbf{H}$ reads

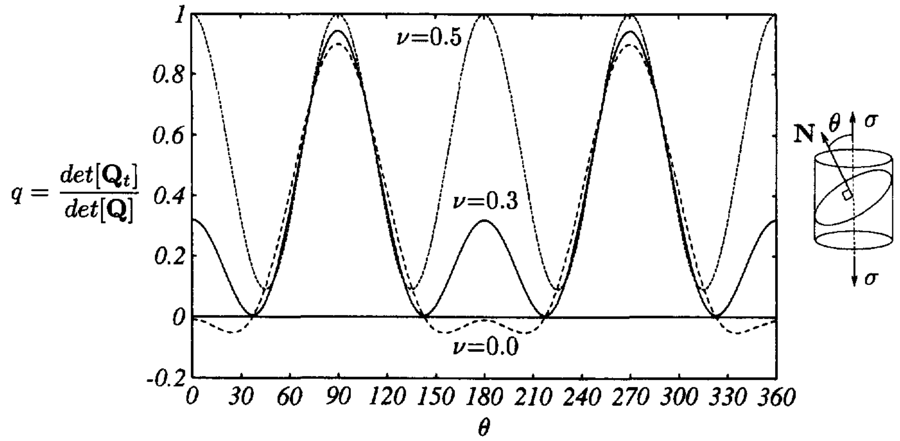

Fig. 6. Influence of Poisson ratio on the localization indicator q for $h_D = h_P = 0.65$ in uniaxial tension.

$$\mathbf{H} = \begin{bmatrix} H_D & -\mathbf{n}_D : \mathbf{E} : \mathbf{n}_P \\ -\mathbf{n}_P : \mathbf{E} : \mathbf{n}_D & H_P \end{bmatrix}. \tag{79}$$

This implies interactive hardening, except for purely volumetric stress states. The localization indicator, eqn (73), reads in this case:

$$q = 1 - \frac{\mathbf{n}_D : \mathbf{E}^N : \mathbf{n}_D / \mathbf{n}_D : \mathbf{E} : \mathbf{n}_D}{(1+h_D)} - \frac{\mathbf{n}_D : \mathbf{E}^N : \mathbf{n}_P / \mathbf{n}_P : \mathbf{E} : \mathbf{n}_P}{(1+h_P)} - \frac{-\det[\mathbf{H}_c^N]}{(1+h_D)(1+h_P)} \tag{80}$$

where h_D, h_P are the normalized hardening variables $h_D = H_D(\mathbf{n}_D : \mathbf{E} : \mathbf{n}_D)^{-1}$ and $h_P = H_P(\mathbf{n}_P : \mathbf{E} : \mathbf{n}_P)^{-1}$, which are $\geqslant -1$ for the P-property of $\bar{\mathbf{H}}$.

6.1. *Loading in uniaxial tension*

The results of localization analysis in uniaxial tension are plotted in Figs 6 and 7. The possible localization directions form a rotational cone around the loading axis, identified by the angle θ between the load axis and the normal $\mathbf{N}$ to the possible discontinuity surface. Comparisons are made with the case when only the plastic dissipation or the scalar damage mechanism is present. For von Mises plasticity a considerable amount of softening is required for localization in a three-dimensional interpretation of uniaxial tension test (Ottosen and Runesson, 1991a), while for scalar damage the onset of localization happens at the limit point (Rizzi *et al.*, 1995). In both cases the critical localization directions for

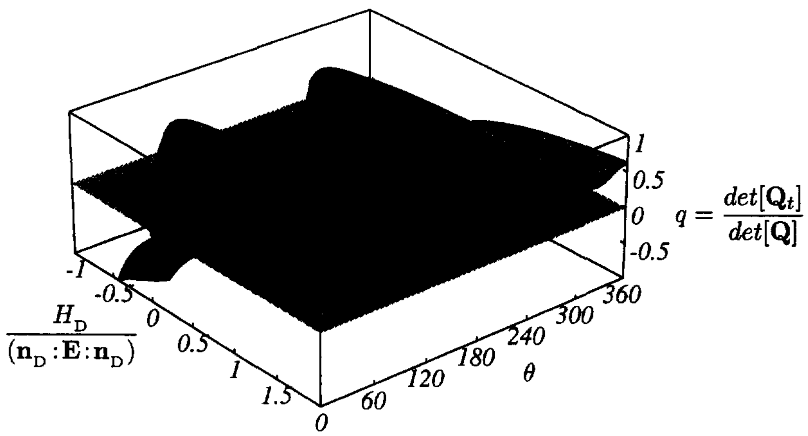

Fig. 7. Discontinuous failure indicator q along the path $h_D = h_P$ in uniaxial tension ($\nu = 0.3$).

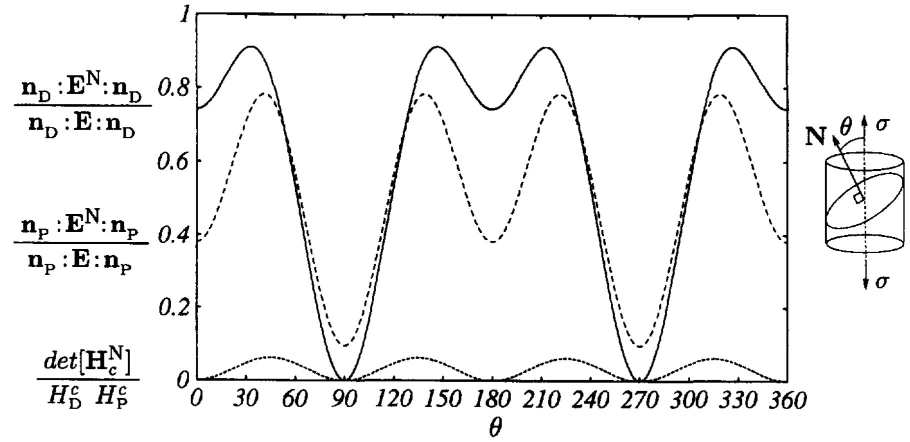

Fig. 8. Values of the quadratic forms associated to $\mathbf{E}^N$ in uniaxial tension ($\nu = 0.3$).

which the failure indicator q is a minimum depend on Poisson's ratio. When both dissipation mechanisms are simultaneously active, localization occurs earlier. In Figs 6 and 7 the choice of the hardening parameters is restricted to the diagonal path $h_D = h_P$. Figure 6 clearly shows the destabilizing effect, since the onset of localization may happen when both mechanisms are still in the hardening range. The q-plots, and in particular the position of the local minima, strongly vary with Poisson's ratio (Fig. 6) and the amount of hardening (Fig. 7).

The acceleration effect does not show up in Fig. 7 as high softening values $h_D \leqslant -0.9$ are cut-off. However, this effect should not be ruled out *a priori*, as it can be seen in Fig. 8 which illustrates the θ-variation of the non-dimensional quantities in eqn (80). All quantities $\mathbf{n}_D : \mathbf{E}^N : \mathbf{n}_D$, $\mathbf{n}_P : \mathbf{E}^N : \mathbf{n}_P$, $\det[\mathbf{H}_c^N]$ are positive or zero. Since $\det[\mathbf{H}_c^N]$ does not vanish identically (if $\nu \neq 0.5$), acceleration is possible for those directions where $\det[\mathbf{H}_c^N] > 0$ and for softening close to the limiting values $h_D = h_P = -1$.

6.2. *Loading in pure shear*

Similar conclusions on the destabilizing effect of combining plasticity and damage can be drawn for the pure shear case (Figs 9 and 10). Localization is possible when both mechanisms are in the hardening regime, $h_D = h_P = 1$, whereas it would occur for zero hardening in von Mises plasticity (Ottosen and Runesson, 1991a) and in the scalar damage context (Rizzi *et al.*, 1995). The locations of the minima do not depend on ν and h_D. Both effects of locking and acceleration are precluded since $\det[\mathbf{H}_c] = \det[\mathbf{H}_c^N] \equiv 0$.

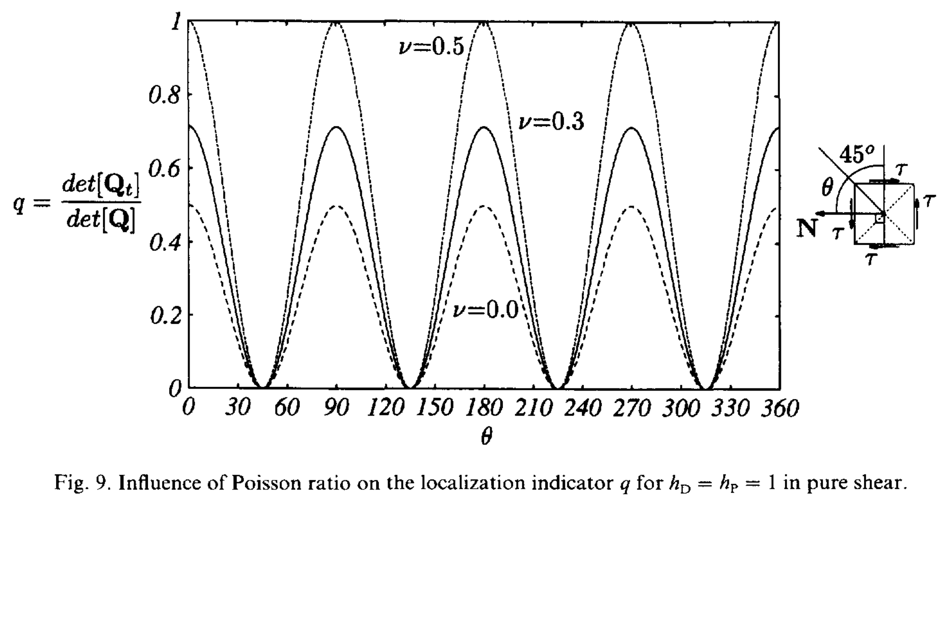

Fig. 9. Influence of Poisson ratio on the localization indicator q for $h_D = h_P = 1$ in pure shear.

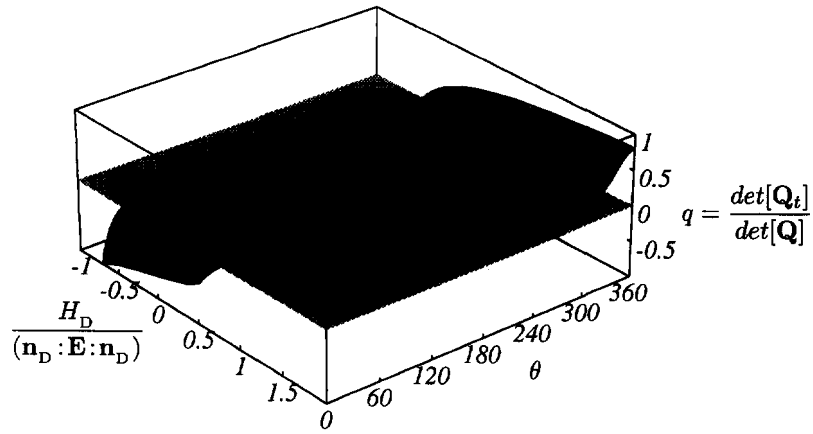

Fig. 10. Discontinuous failure indicator q along the path $h_D = h_P$ in pure shear ($\nu = 0.3$).

7. CLOSING REMARKS

The inelastic rate models analysed in this paper are intended for describing incremental deformation processes in multi-dissipative materials. The classes of material phenomena include multiple yielding modes at corners in non-associative elastoplasticity, damage understood as stiffness degradation, and combinations of damage and plastic flow. The contents and conclusions of the present study may be summarized as follows:

(a) The tangent operators, i.e. the instantaneous stiffness tensor and the localization (acoustic) tensor, result from multiple rank-one modifications (updates) of the elastic reference tensors (cf. Sections 2 and 3). This circumstance and the systematic recourse to spectral properties of rank-one updates provide a unified methodology, centered on the eigensolution of identity matrix updates. They lead to the analytical evaluation of failure indicators for loss of stability and uniqueness, and for strain localization, concomitant with the onset of discontinuities in the strain rates. Both weak and strong failure criteria were examined (Section 3), which play a distinct role when non-symmetric operators are considered.

(b) The spectral approach was applied in Section 4 to the familiar single-dissipation models (i.e. to smooth boundaries of the elastic domain). On this basis four different failure criteria were systematically discussed, re-deriving known results of material instability and localization theory, and expounding on their interrelationships and similarities.

(c) For double-dissipation models closed-form solutions were derived in Section 5 which illustrated destabilizing effects in hardening materials, as well as spurious locking (stiffness increase) and acceleration (wave speed increase) in softening formulations. In this context, critical threshold conditions were developed for the hardening–softening parameters.

(d) The focus of Section 6 was double-dissipation models which combine von Mises plasticity and scalar damage. Continuous and discontinuous bifurcation conditions were investigated in some detail with specific reference to uniaxial and pure shear tests in three-dimensions. It was found that the localization direction is strongly influenced by Poisson's ratio and the hardening parameters in uniaxial tension, not in pure shear. As a consequence of the simultaneity of dissipation processes, diffuse and discontinuous failure modes were possible when both dissipation mechanisms were still in the hardening range.

(e) The interpretation of tangent operators in terms of rank-one updates and its analytical consequences suggest the study of the following two conjectures, (i) that the most critical condition for the onset of failure occurs when all possible dissipation modes are active, and (ii) that localization is critical when simultaneous loading of all possible active dissipation mechanisms takes place on each side of the weak discontinuity surface.

Acknowledgements—G. M. and E. R. acknowledge with thanks a MURST research grant (so-called 40%), mechanics of materials group, at Politecnico di Milano; K. W. and E. R. thank Dr Ken Chong for the partial financial support under NSF-grant MSS-9203047 at the University of Colorado, Boulder.

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APPENDIX A. RANK-ONE UPDATES AND RELEVANT SPECTRAL ANALYSIS

Consider the following two square matrices of order m, which are in general non-symmetric if $\mathbf{a}_i \neq \mathbf{b}_i$ for some i:

$$\mathbf{U}_n = \sum_{i=1}^{n} \mathbf{a}_i \otimes \mathbf{b}_i, \quad \text{and} \quad \mathbf{V}_n = \mathbf{I} + \mathbf{U}_n. \tag{A.1}$$

Here $\mathbf{a}_i$, with $i = 1, 2, \ldots, n$, are n linearly independent vectors spanning $\mathscr{R}^m$, and so are $\mathbf{b}_i$. Summation convention on repeated indices is not used in this appendix. Each matrix $\mathbf{a}_i \otimes \mathbf{b}_i$ can be easily shown to have rank 1 and, hence, when added or subtracted to a matrix, will be referred to as "rank-one update" of the matrix. Thus, matrix $\mathbf{V}_n$ can be conceived as generated from the identity matrix $\mathbf{I}$ of order m through n successive rank-one updates.

The rank r of $\mathbf{U}_n$ cannot exceed n (nor m, of course), i.e.:

$$m \geqslant r[\mathbf{U}_n] \leqslant n. \tag{A.2}$$

This is a consequence of the inequality $r[\mathbf{A}+\mathbf{B}] \leqslant r[\mathbf{A}]+r[\mathbf{B}]$ which holds for any pair of square matrices $\mathbf{A}$ and $\mathbf{B}$ (Mirsky, 1955) since each update $\mathbf{a}_i \otimes \mathbf{b}_i$ has rank 1. Thus, when $n < m$, $r[\mathbf{U}_n]$ is at most n, and matrix $\mathbf{V}_n$ can be conceived as through a single "rank-n update" of the identity matrix $\mathbf{I}$.

The eigenvalue problem for matrix $\mathbf{U}_n$ reads:

$$\mathbf{U}_n \cdot \mathbf{x} = \sum_{i=1}^{n} (\mathbf{b}_i \cdot \mathbf{x})\mathbf{a}_i = u\mathbf{x}. \tag{A.3}$$

In view of inequalities (A.2), if $n < m$, no more than n eigenvalues u_k may be different from zero. From eqn (A.3) let eigenvectors be sought in the form

$$\mathbf{x} = \sum_{j=1}^{n} \alpha_j \mathbf{a}_j \tag{A.4}$$

where α_i are unknown coefficients. Substituting eqn (A.4) into (A.3), we obtain:

$$\sum_{i=1}^{n} \left(\sum_{j=1}^{n} [\mathbf{b}_i \cdot \mathbf{a}_j - u\,\delta_{ij}]\alpha_j \right) \mathbf{a}_i = \mathbf{0} \tag{A.5}$$

which turns out to be satisfied if α_j ($j = 1, 2, \ldots, n$) are such that all the n expressions in the round brackets of (A.5) vanish (for $i = 1, 2, \ldots, n$), i.e. if the vector $\boldsymbol{\alpha}$ and the scalar u are solutions of the following eigenproblem:

$$\sum_{j=1}^{n} [\mathbf{b}_i \cdot \mathbf{a}_j - u\delta_{ij}]\alpha_j = 0 \; (i = 1, 2, \ldots, n) \quad \text{or} \quad \mathbf{Z} \cdot \boldsymbol{\alpha} = u\boldsymbol{\alpha} \tag{A.6}$$

having introduced the $(m \times n)$ matrices $\mathbf{A} \equiv [\mathbf{a}_j]$, $\mathbf{B} \equiv [\mathbf{b}_j]$, and the square $(n \times n)$ matrix $\mathbf{Z} \equiv [\mathbf{b}_i \cdot \mathbf{a}_j] = \mathbf{B}^{\mathrm{T}} \cdot \mathbf{A}$ ($i,j = 1, 2, \ldots, n$). The problem (A.6) in $\mathscr{R}^n$ is then associated to the original problem (A.3) in $\mathscr{R}^m$ through the following links:

(a) Both $\mathbf{U}_n$ and $\mathbf{Z}$ are symmetric and, hence, have real eigenvalues if $\mathbf{B} = \mathbf{A}$.

(b) If $n > m$ (i.e. if the number n of rank-one updates exceeds the original dimensionality m), the non-zero eigenvalues of $\mathbf{Z}$ do not exceed m. This latter remark stems from the general inequality $r[\mathbf{B}^{\mathrm{T}} \cdot \mathbf{A}]$

$\leqslant \min\{r[\mathbf{A}], r[\mathbf{B}]\}$ for any $\mathbf{A}$ and $\mathbf{B}$ (Mirsky, 1955), and from the circumstance that $r[\mathbf{A}] \leqslant m$ and $r[\mathbf{B}] \leqslant m$, which renders $r[\mathbf{Z}] \leqslant m$.

The eigenvalue problem of the matrix $\mathbf{V}_n$ associated to the m-identity matrix modified by n rank-one updates, as defined in eqn (A.1),

$$\mathbf{V}_n \cdot \mathbf{y} = v\mathbf{y} \tag{A.7}$$

is related to the eigenvalue problem (A.3) for $\mathbf{U}_n$. In fact, the eigenvalues v_k and eigenvectors $\mathbf{y}_k$ of $\mathbf{V}_n$ are easily seen to be related to the eigenpairs u_k, $\mathbf{x}_k$ of $\mathbf{U}_n$ by:

$$v_k = 1 + u_k, \quad \mathbf{y}_k = \mathbf{x}_k; \quad k = 1, 2, \ldots, m. \tag{A.8}$$

The special cases of one ($n = 1$) and two ($n = 2$) rank-one updates are the focus below.

A.1. *Rank-one update of the identity matrix*

For $n = 1$ we have $\mathbf{U}_1 = \mathbf{a} \otimes \mathbf{b}$ and $\mathbf{Z} = \mathbf{a} \cdot \mathbf{b} = Z$ reduces to a scalar. The relevant eigensolutions in (A.3) and (A.6) are $u_1 = \mathbf{a} \cdot \mathbf{b}$, $\mathbf{x}_1 = \mathbf{a}$, and $u_1 = \mathbf{a} \cdot \mathbf{b}$, $\alpha_1 = 1$, respectively. The other $(m-1)$ eigenvalues of $\mathbf{U}_1$ are zero and correspond to eigenvectors $\mathbf{x}_k$ orthogonal to $\mathbf{b}$, since $\mathbf{b} \cdot \mathbf{x}_k = 0$, as can be noticed from eqn (A.3). For the modified identity matrix $\mathbf{V}_1$, in view of eqn (A.8), we obtain the eigenvalue spectrum, $v_1 = 1 + u_1$, $v_k = 1$ $(k = 2, 3, \ldots, m)$ and the associated eigenvectors $\mathbf{y}_k = \mathbf{x}_k$. Spectral analysis of the m-matrix $\mathbf{V}_1$ provides the determinant in the form:

$$\det[\mathbf{V}_1] = \det[\mathbf{I} + \mathbf{a} \otimes \mathbf{b}] = 1 + \mathbf{a} \cdot \mathbf{b}. \tag{A.9}$$

A.2. *Two rank-one updates of the identity matrix*

For $n = 2$, $\mathbf{U}_2 = \mathbf{a}_1 \otimes \mathbf{b}_1 + \mathbf{a}_2 \otimes \mathbf{b}_2$, and the square matrix $\mathbf{Z}$ has order 2 and, hence, its eigenvalues can be expressed in closed form (by solving a second order equation):

$$\begin{aligned} u_{1,2} &= \tfrac{1}{2}\left((\mathbf{a}_1 \cdot \mathbf{b}_1 + \mathbf{a}_2 \cdot \mathbf{b}_2) \pm \sqrt{(\mathbf{a}_1 \cdot \mathbf{b}_1 + \mathbf{a}_2 \cdot \mathbf{b}_2)^2 - 4(\mathbf{a}_1 \cdot \mathbf{b}_1 \mathbf{a}_2 \cdot \mathbf{b}_2 - \mathbf{a}_1 \cdot \mathbf{b}_2 \mathbf{a}_2 \cdot \mathbf{b}_1)}\right) \\ &= \tfrac{1}{2}\left((\mathbf{a}_1 \cdot \mathbf{b}_1 + \mathbf{a}_2 \cdot \mathbf{b}_2) \pm \sqrt{(\mathbf{a}_1 \cdot \mathbf{b}_1 - \mathbf{a}_2 \cdot \mathbf{b}_2)^2 + 4\mathbf{a}_1 \cdot \mathbf{b}_2 \mathbf{a}_2 \cdot \mathbf{b}_1}\right). \end{aligned} \tag{A.10}$$

The closed form solution of the corresponding eigenvectors, $\boldsymbol{\alpha}_{1,2}$ of $\mathbf{Z}$, follows from eqn (A.6). In view of eqn (A.4), the eigenvectors of $\mathbf{U}_2$ associated with the eigenvalues $u_{1,2}$ compute

$$\mathbf{x}_k = \alpha_1^k \mathbf{a}_1 + \alpha_2^k \mathbf{a}_2, \quad k = 1, 2. \tag{A.11}$$

Finally, the two eigenvalues of $\mathbf{V}_2$ which are not necessarily unitary, i.e. $v_k = 1 + u_k$ $(k = 1, 2)$, provide an explicit expression for the determinant of the two rank-one modifications of the m-identity matrix, i.e.

$$\det[\mathbf{V}_2] = \det[\mathbf{I} + \mathbf{a}_1 \otimes \mathbf{b}_1 + \mathbf{a}_2 \otimes \mathbf{b}_2] = (1 + \mathbf{a}_1 \cdot \mathbf{b}_1)(1 + \mathbf{a}_2 \cdot \mathbf{b}_2) - \mathbf{a}_1 \cdot \mathbf{b}_2 \mathbf{a}_2 \cdot \mathbf{b}_1. \tag{A.12}$$

This expression coincides with the "formula for the perturbation of a determinant" established by Pearson (1969).

For $\mathbf{a}_i = \mathbf{b}_i$, $i = 1, 2$ (i.e. $\mathbf{A} = \mathbf{B}$), the discriminant in eqn (A.10) remains positive, i.e. $\Delta = (\mathbf{a}_1 \cdot \mathbf{a}_1 - \mathbf{a}_2 \cdot \mathbf{a}_2)^2 + 4(\mathbf{a}_1 \cdot \mathbf{a}_2)^2 > 0$, so that complex eigenvalues are ruled out. This is expected since for $\mathbf{A} = \mathbf{B}$, matrices $\mathbf{U}_2$ and $\mathbf{Z}$ are symmetric. However, when $\mathbf{b}_i = \rho_i \mathbf{a}_i$, $i = 1, 2$ with $\rho_1 \neq \rho_2$ (i.e. when vectors $\mathbf{a}_i$ and $\mathbf{b}_i$ are coaxial but not equal), the m-matrix $\mathbf{U}_2$ is still symmetric, but $\mathbf{Z}$ is no longer symmetric. Nevertheless, its eigenvalues in eqn (A.10) are still real. In fact, denoting by a_1, a_2 and ϕ the Euclidean norm of $\mathbf{a}_1$, $\mathbf{a}_2$ and the angle between $\mathbf{a}_1$ and $\mathbf{a}_2$, respectively, the two alternative expressions of the discriminant Δ in eqn (A.10) read:

$$\Delta = (\rho_1 a_1^2 + \rho_2 a_2^2)^2 - 4\rho_1 \rho_2 a_1^2 a_2^2 \sin^2\phi = (\rho_1 a_1^2 - \rho_2 a_2^2)^2 + 4\rho_1 \rho_2 a_1^2 a_2^2 \cos^2\phi > 0 \tag{A.13}$$

which turns out to remain positive for any non-zero values of ρ_1, ρ_2.

APPENDIX B. ON P-MATRICES

Some linear algebra notions on P-matrices of use in the paper are gathered below.

Definition

The "principal submatrix" $\mathbf{A}^\alpha$ of matrix $\mathbf{A} \in \mathscr{R}^{n \times n}$ is the matrix whose entries lie in rows and columns of $\mathbf{A}$ indexed by sets $\alpha \subseteq \{1, 2, \ldots, n\}$ (i.e. $\mathbf{A}^\alpha$ is the submatrix of $\mathbf{A}$ whose diagonal is part of the diagonal of $\mathbf{A}$). The determinant of $\mathbf{A}^\alpha$ is called a "principal minor" of $\mathbf{A}$.

Definition

The matrix $\mathbf{A} \in \mathscr{R}^{n \times n}$ "reverses the sign" of the vector $\mathbf{x} \in \mathscr{R}^n$ if

$$x_i(\mathbf{A} \cdot \mathbf{x})_i \leqslant 0 \quad \forall\, i = 1, 2, \ldots, n.$$

Definition

A matrix $\mathbf{A} \in \mathscr{R}^{n \times n}$ is said to be a "P-matrix" when all its principal minors are positive.

The main properties of P-matrices are expressed by the following theorem (Cottle, 1992).

Let $\mathbf{A} \in \mathscr{R}^{n \times n}$. The following statements are equivalent:

(a) $\mathbf{A}$ is a P-matrix.

(b) $\mathbf{A}$ does not reverse the sign of any vector, except the zero vector, i.e.

$$x_i(\mathbf{A} \cdot \mathbf{x})_i \leqslant 0 \quad \forall\, i = 1, 2, \ldots, n. \quad \text{implies} \quad \mathbf{x} = \mathbf{0}.$$

(c) All real eigenvalues of $\mathbf{A}$ and its principal submatrices are positive.

Remarks

The set of all symmetric P-matrices coincides with the set of all symmetric positive definite matrices.

The set of all positive definite matrices is a subset of the P-matrices set, i.e. there are (non-symmetric) P-matrices which are not positive definite, whereas every positive definite matrix is a P-matrix.